Improved EXIT Analysis for Turbo Decoding

Tsung-Ru Tsai, Der-Feng Tseng, *Member, IEEE*, Yunghsiang S. Han, *Fellow, IEEE*, and Hung-Ta Pai, *Member, IEEE*

***Abstract*—Extrinsic information transfer (EXIT) chart is a useful tool for analyzing iterative decoding behavior. One crucial challenge in realizing it is that a precise estimate of the probability density function (PDF) of the extrinsic information at the output of a soft-input soft-output decoder must be made to obtain mutual information. To achieve this, we employed a kernel density estimator (KDE), as opposed to using the traditional histogram estimator, when a fairly small interleaver size is available. Simulation results showed that the KDE approach outperformed the histogram method for estimating the PDF in the EXIT band analysis in terms of bit error rate prediction.**

***Index Terms*—Turbo code, histogram, kernel density estimation, EXIT chart.**

I. Introduction

SINCE 1993, the turbo code [1] has become a popular error correcting code (ECC) in communications systems due to its powerful error correcting capability. Indeed, a turbo decoder, composed of two soft-input soft-output (SISO) decoders which iteratively exchange soft information to improve the decoding robustness against noise, can be computationally intensive.

Extrinsic information transfer (EXIT) chart, introduced in [2], serves as a convenient tool that, without extensive computer simulations, provides expedited information about the signal-to-noise ratio (SNR) associated with the bit error rate (BER) waterfall region upon selected parameters of codes such as generator matrices and code rate. Note that the Gaussian assumption for the a-posteriori log likelihood (LLR) sequence is no longer valid for channels other than additive white Gaussian noise (AWGN) [2], prompting the need for the probability density function (PDF) estimation, which is also a critical step for deriving the mutual information (MI) between the information bit and extrinsic information at outputs of SISO decoders on the EXIT chart. Further, the EXIT function relies on its independence of the interleaver size, restricting its applications in standards such as [3]. Alternatively, the SNR band charts [4] provide an analytical tool for the interleaver-length dependent turbo codes by taking advantage of the bimodal histogram of the extrinsic information for each information bit over an ensemble of all channel realizations.

In this paper, we employ kernel density estimation (KDE) to estimate the PDF of the extrinsic LLR sequence in the EXIT analysis for short-length turbo codes over Rayleigh flat-fading channel. Indeed, the smoothing characteristic of the kernel function for random samples of much smaller size by virtue of KDE is its primary advantage against the PDF estimation bias encountered in histogram due to fixed bin width constraint. The simulation outcome confirmed the superiority of using KDE on estimating PDF for the EXIT analysis in terms of the match between EXIT bands and decoding trajectories.

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T.-R. Tsai, D.-F. Tseng and Y. S. Han are with the Department of Electrical Engineering, National Taiwan University of Science and Technology, Taipei, Taiwan, R.O.C. (e-mail: dtseng@mail.ntust.edu.tw).

H.-T. Pai is with the Graduate Institute of Communication Engineering, National Taipei University, Taipei, Taiwan, R.O.C.

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II. EXIT Chart and Histogram Estimation

In the pioneering work [1], the system architecture of a turbo code was constituted with two recursive systematic convolutional (RSC) codes in a parallel concatenated format. For instance, a bit stream $d_k \in \{0,1\}$ with k the data bit time index, is input to the RSC encoders, where equal probabilities for bits 0 and 1 are assumed. Considering a punctured rate-$1/2$ code [1], the transmitted signal sequence z_j is thus denoted by

$$z_j = \begin{cases} x_{j/2} & j = 0,2,4,6,\ldots \\ y_{1,(j-1)/2} & j = 1,5,\ldots \\ y_{2,(j-1)/2} & j = 3,7,\ldots, \end{cases}$$

where j is the code bit time index. The signal $x_j \in \{-1,1\}$ is a binary phase shift keying (BPSK) modulation of d_k, while $y_{i,j} \in \{-1,1\}$ and $i = 1,2$ are BPSK symbols corresponding to redundancy. Assuming that the signaling undergoes a Rayleigh flat-fading channel, the received signal can be represented by

$$r_j = h_j z_j + n_j,$$

where the fading gain h_j is a complex Gaussian random variable with zero mean and unity variance and n_j is modeled as a complex-valued circularly symmetric zero-mean white Gaussian with variance σ_n^2.

To decode data, the receiver first de-multiplexes r_j such that the extrinsic information, derived from the BCJR algorithm and implemented at SISO decoders, is exchanged iteratively prior to recursive termination [1] .

A. The EXIT Chart

As an analysis tool, the EXIT chart relates the MI between an information bit X and its a-priori information, in terms of LLR, to the MI between X and its extrinsic information, also in terms of LLR. Indeed, as the LLR at the output of one SISO decoder at current iteration is fed, after de-interleaving/interleaving, into another as input for further processing at the next iteration, an evolution between two MI values, i.e., I_A and I_E, can be characterized by a transfer function $I_E = T(I_A)$, and based on which one can thus depict the decoding trajectory on the EXIT chart. For BPSK

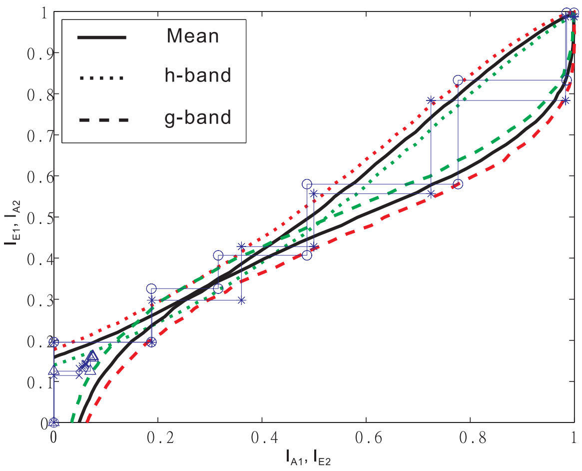

Fig. 1. The EXIT band chart by histogram. SNR = 2.47dB.

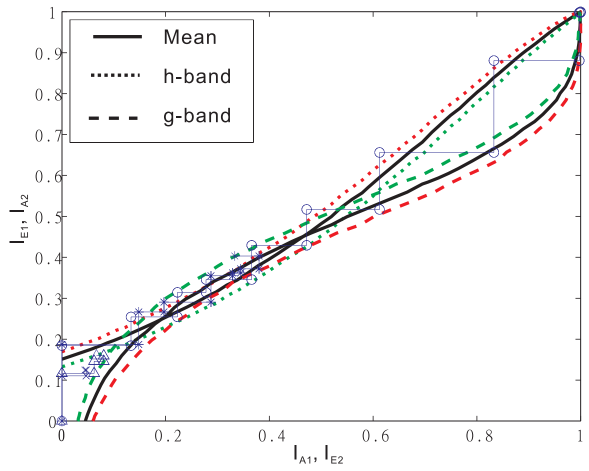

Fig. 2. The EXIT band chart by KDE. SNR = 2.47dB.

modulation, the MI $I_\xi(\xi \in A, E)$ of the LLRs are of the same form as [5]:

$$I_\xi = \frac{1}{2}\sum_{x=1,-1}\int_{-\infty}^{\infty} p_\xi(\alpha|X=x) \cdot \log_2 \frac{2p_\xi(\alpha|X=x)}{p_\xi(\alpha|X=1)+p_\xi(\alpha|X=-1)} d\alpha, \quad (1)$$

where A and E correspond to the a-priori LLR sequence and extrinsic LLR output sequence, respectively. To facilitate the I_A computation, the statistical distribution $p_A(\alpha|X)$ is generally assumed to be Gaussian [2] with mean $\mu_a = \pm\sigma_a^2/2$, where σ_a^2 is the variance and the sign of μ_a depends on the value of X, provided that consistency property [5] is satisfied. Nevertheless, estimation for $p_E(\alpha|X)$ to measure I_E is inevitable for channels other than AWGN and is generally invoked via Monte Carlo simulations [2].

B. The Histogram Estimation

Histogram is a popular approach to estimating $p_E(\alpha|X)$, where an equally spaced bin width h is normally employed. A useful metric for assessment of PDF estimation, the integrated mean squared error (IMSE), is measured as

$$\frac{1}{Lh} + \frac{h^2}{12}\int_{-\infty}^{\infty} (dp_E(\alpha|X=x)/d\alpha)^2 d\alpha, \quad (2)$$

where L is the interleaver size. The optimal bin width h^* to minimizing (2) is [6]:

$$h^* = 1.82\left(L\int_{-\infty}^{\infty}(dp_E(\alpha|X=x)/d\alpha)^2 d\alpha\right)^{-\frac{1}{3}}, \quad (3)$$

which also leads to the minimum IMSE as

$$0.8255L^{-\frac{2}{3}}\left(\int_{-\infty}^{\infty}(dp_E(\alpha|X=x)/d\alpha)^2 d\alpha\right)^{\frac{1}{3}}. \quad (4)$$

Indeed, the minimum IMSE decreases with the interleaver size L to an order of 2/3 and is also subject to some function of the derivative of the true PDF, which is however unknown in advance.

III. Kernel Density Estimation

In contrast to histogram, the bin element for the PDF estimation by KDE is replaced by a kernel function $K(t)$. The estimator for $p_E(\alpha|X=x)$ is thus:

$$\hat{p}_E(\alpha|X=x) = \frac{1}{L\omega_K}\sum_{i=1}^{L} K\left(\frac{\alpha - L_i}{\omega_K}\right),$$

where ω_K is the window size for scaling $K(t)$ and L_i is the i-th sample of LLR sequence. It is noteworthy that KDE is flexible in terms of discretization of $\hat{p}_E(\alpha|X=x)$, for its independence of L as well as ω_K, whereas the histogram estimation is subject to a fixed bin width. Similarly, one can obtain the IMSE of KDE [7] as

$$\text{IMSE} = \frac{1}{L\omega_K}\int_{-\infty}^{\infty} K^2(t)dt + \frac{1}{4}\omega_K^4 k_2^2 \int_{-\infty}^{\infty}\left(d^2p_E(\alpha|X=x)/d\alpha^2\right)^2 d\alpha, \quad (5)$$

where $k_2 = \int_{-\infty}^{\infty} t^2K(t)dt$. Minimizing (5) with respect to ω_K can lead to the optimal window width ω_K^* as

$$\left\{k_2^2 L\int_{-\infty}^{\infty}\left(d^2p_E(\alpha|X=x)/d\alpha^2\right)^2 d\alpha \Big/ \int_{-\infty}^{\infty} K^2(t)dt\right\}^{-\frac{1}{5}},$$

which is accompanied by the minimum IMSE value as

$$\frac{5}{4L^{\frac{4}{5}}}C_K\left\{\int_{-\infty}^{\infty}\left(d^2p_E(\alpha|X=x)/d\alpha^2\right)^2 d\alpha\right\}^{\frac{1}{5}},$$

where $C_K = \left\{\int_{-\infty}^{\infty}\sqrt{k_2}K^2(t)dt\right\}^{4/5}$. Note that the minimized IMSE by KDE is inversely proportional to interleaver size L to an order of $4/5$, up from $2/3$ when compared with the histogram, indicating that the estimation of conditional PDF $P_E(\alpha|X)$ by means of KDE is less susceptible to a downsized L. As a rule of thumb, we use the maximal smoothing principle to obtain ω_K^* under the realistic situation in which the knowledge of true $p_E(\alpha|X)$ is absent and impossible to acquire. Along with this optimal window width to minimum IMSE is the Epanechnikov kernel function:

$$K_e(t) = \begin{cases} \frac{3}{4\sqrt{5}}\left(1-\frac{1}{5}t^2\right) & -\sqrt{5} \le t \le \sqrt{5} \\ 0 & \text{otherwise} \end{cases}. \quad (6)$$

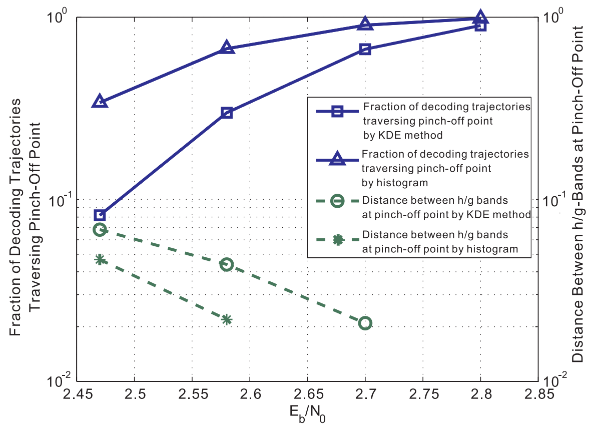

Fig. 3. Fraction of decoding trajectories navigating past pinch-off point and at which the distance between h/g-bands.

IV. Simulation Results

A rate-1/2 parallel concatenated turbo code with generator matrices [7 5] and [147 117] in octal form, respectively, and an interleaver size $L = 5,000$ compliant with [3] is employed over a Rayleigh flat-fading channel in the EXIT analysis, where the optimal bin width (3) is used for histogram while a Gaussian window is employed as the kernel function due to its trivial performance gap with (6)[7], also attested in our experiments. We plot the EXIT band charts, akin to the SNR bands in [4], by histogram and KDE methods for P_E at 2.47dB SNR in Figs. 1 and 2, respectively. As one can observe, the MI, averaged out over 250 independent channel runs by KDE, associated with solid lines in Fig. 2, induce a closed tunnel, whereas by histogram the tunnel, as seen in Fig. 1, is about to open. Moreover, even though h/g-band bounds in both plots intersect, a majority of decoding trajectories by histogram, with steps alike drawn in Fig. 1, are likely to arrive at the upper rightmost corner only after a few iterations, while a fair number of trajectories by KDE in Fig. 2 come across an impasse to advance. Indeed, from our extensive simulations, the pinch-off points in the EXIT chart by the KDE and histogram methods are associated with SNRs of 2.58dB and 2.47dB, respectively.

A logarithmic plot of the fraction of decoding trajectories that navigate past the pinch-off point in the EXIT chart is depicted, as a function of SNR, in Fig. 3. One can observe that, regardless of the approach to measuring MI, it is not until 2.8dB SNR that almost all of trajectory snapshots can traverse pinch-off points, a phenomenon indeed dictated by channel conditions. The distance (in logarithm) between the h- and g-bands at the pinch-off point for both KDE and histogram methods is also plotted in Fig.3. One can view that the distance induced by KDE declines with SNR at a slower pace than that by its histogram counterpart, indicating the KDE method more precisely predicts whether the EXIT chart tunnel is considered to be open or closed, which is corroborated by the Monte Carlo simulation result, omitted here due to space limitation, which reveals that under this scenario a 10^{-4} BER is not achieved until 2.96dB SNR, a point nevertheless beyond the SNR range in Fig.3.

V. Conclusion

The PDF estimator by KDE for measuring MI in the EXIT chart for short-length turbo codes over Rayleigh flat-fading channels is proposed. The improved PDF estimation in terms of IMSE yields more accurate EXIT bands in comparison with its histogram counterpart, which in turn, through justification of extensive simulation results, offers better assessment in critical SNR regime to engineers engaged in commercial standards.

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