Chapter 2: Basic Concepts of Probability Theory[1]

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[1] Modified from the lecture notes by Prof. Mao-Ching Chiu

Specifying random experiments

1. State an experiment procedure.
2. Define a set of one or more measurements or observations.

Example:

1. Toss a coin three times and
2. note the number of heads.

Sample Space

S: Set of all possible outcomes.

Example: Select a ball from an urn that contains balls numbered 1 to 50. Note the number of balls. $S = \{1, 2, \ldots, 50\}$.

- Discrete sample space: S is countable, e.g., it contains integers.
- Continuous sample space: S is not countable, e.g., it contains real number.

- Pick a number at random between zero and one. $S = [0, 1]$.
- Cartesian product: Toss a coin three times. $S_3 = S \times S \times S$.

Events

Events: Outcome satisfies certain conditions.

Example:

- Determine the value of a voltage waveform at time t.
- $S = \{v : -\infty < v < \infty\} = (-\infty, \infty)$.
- Event E: voltage outcome ψ is negative.
- $E = \{\psi : -\infty < \psi < 0\}$.

Axioms of Probability

Let S be the sample space. Assign to each event A a number $P[A]$, probability of A, that satisfies the axioms.

Axiom I: $0 \leq P[A]$.

Axiom II: $P[S] = 1$.

Axiom III: If $A \cap B = \emptyset$, then $P[A \cup B] = P[A] + P[B]$. **Axiom III'**: If $A_1, A_2, \ldots,$ is a sequence of events such that

$A_i \cap A_j = \emptyset$ for all $i \neq j$, then

$$P\left[\bigcup_{k=1}^{\infty} A_k\right] = \sum_{k=1}^{\infty} P[A_k].$$

Corollary 1 $P[A^c] = 1 - P[A]$

Corollary 2 $P[A] \leq 1$

Corollary 3 $P[\emptyset] = 0$

Corollary 4 *If* $A_1, A_2, \ldots, A_n$ *are pairwise mutually exclusive, then*

$$P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{k=1}^{n} P[A_k] \qquad \textit{for } n \geq 2.$$

Corollary 5

$$P[A \cup B] = P[A] + P[B] - P[A \cap B].$$

$\rightarrow \quad P[A \cup B] \leq P[A] + P[B].$

Corollary 6

$$P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{j=1}^{n} P[A_j] - \sum_{j<k} P[A_j \cap A_k] + \ldots (-1)^{n+1} P[A_1 \cap \ldots \cap A_n].$$

Corollary 7 *If* $A \subset B$, *then* $P[A] \leq P[B]$.

Discrete Sample Space

$S = \{a_1, a_2, \ldots, a_n\}$

Event $B = \{a'_1, a'_2, \ldots, a'_m\}$

$$\begin{aligned} P[B] &= P[\{a'_1, a'_2, \ldots, a'_m\}] \\ &= P[\{a'_1\}] + P[\{a'_2\}] + \ldots + P[\{a'_m\}] \end{aligned}$$

Equally likely outcomes

- $S = \{a_1, \ldots, a_n\}$
- $P[\{a_1\}] = P[\{a_2\}] = \cdots = P[\{a_n\}] = 1/n$
- $B = \{a'_1, \ldots, a'_k\}$
- $P[B] = P[\{a'_1\}] + \ldots + P[\{a'_k\}] = k/n$

Example: An urn contains 10 numbered balls. $S = \{0, 1, \ldots, 9\}$. Assume $P[\{0\}] = P[\{1\}] = \ldots = P[\{9\}] = 1/10$. Find the probability of the following events:

$A =$ "number of ball selected is odd"

$B =$ "number of ball selected is multiple of 3"

$C =$ "number of ball selected is less than 5"

$A \cup B$

$A \cup B \cup C$

Sol: $A = \{1, 3, 5, 7, 9\}$, $B = \{3, 6, 9\}$, $C = \{0, 1, 2, 3, 4\}$

$\rightarrow \quad P[A] = 5/10$, $P[B] = 3/10$, $P[C] = 5/10$

$$\begin{aligned} P[A \cup B] &= P[\{1, 3, 5, 6, 7, 9\}] = 6/10 \\ &= P[A] + P[B] - P[A \cap B] = 5/10 + 3/10 - 2/10 = 6/10 \end{aligned}$$

Continuous Sample Space

Example: measure a voltage or current in a circuit.

Example: Pick a number x at random between zero and one. Suppose that outcomes of $S = [0, 1]$ are equally likely.

$P[[0, 1/2]] = 1/2 \qquad P[[1/2, 1]] = 1/2$

$P[[a, b]] = (b - a) \quad \text{for } 0 \leq a \leq b \leq 1$

$P[\{1/2\}] = 0$

Example: Life time of a computer memory ship.
"The proportion of chips whose life time exceeds t decreases exponentially at a rate α."
$S = (0, \infty)$.

$$P[t, \infty] = e^{-\alpha t} \qquad t > 0$$

Axiom I is satisfied since $e^{-\alpha t} \geq 0$ for $t > 0$.
Axiom II is satisfies since $P[S] = P[(0, \infty)] = 1$.
Since

$$P[(r, \infty)] = P[(r, s]] + P[(s, \infty)]$$

We have

$$P[(r, s]] = P[(r, \infty)] - P[(s, \infty)] = e^{-\alpha r} - e^{-\alpha s}.$$

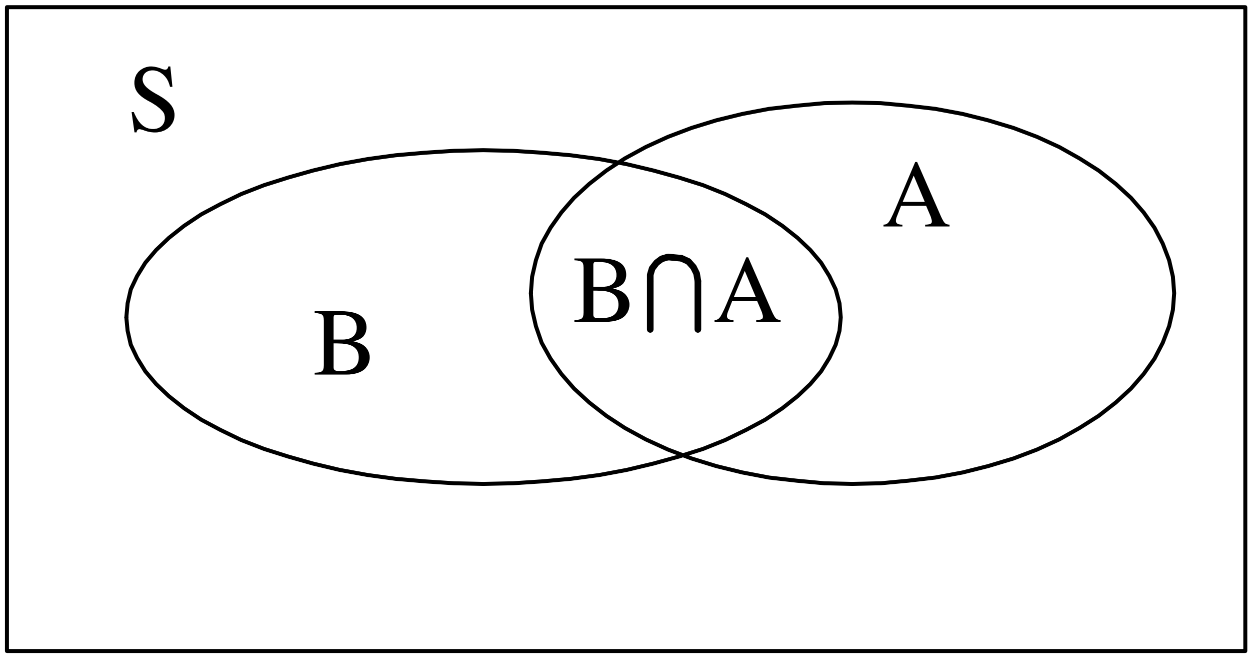
Exercise: Pick two numbers x and y at random between zero and one. $A = \{x > 0.5\}$, $B = \{y > 0.5\}$, $C = \{x > y\}$. Find $P[A]$, $P[B]$, and $P[C]$.

Conditional Probability

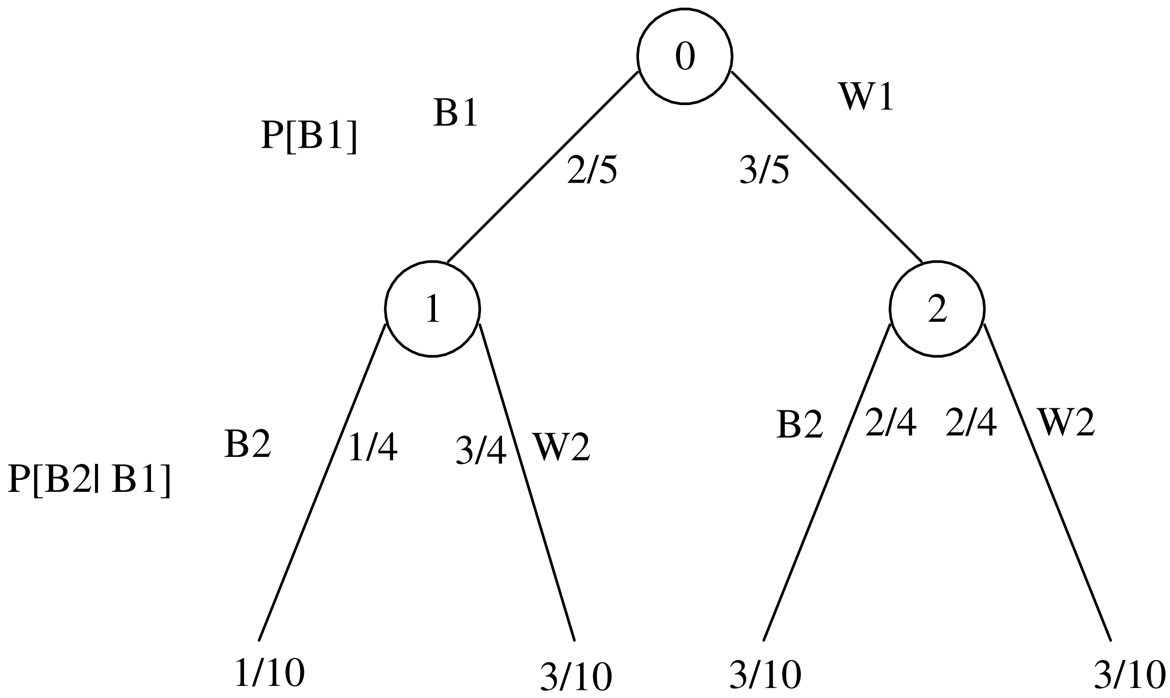
$P[A|B]$: probability of event A given that event B has occurred.

$$P[A|B] = \frac{P[A \cap B]}{P[B]} \qquad \text{for } P[B] > 0$$

Sample space has been reduced from S to B.

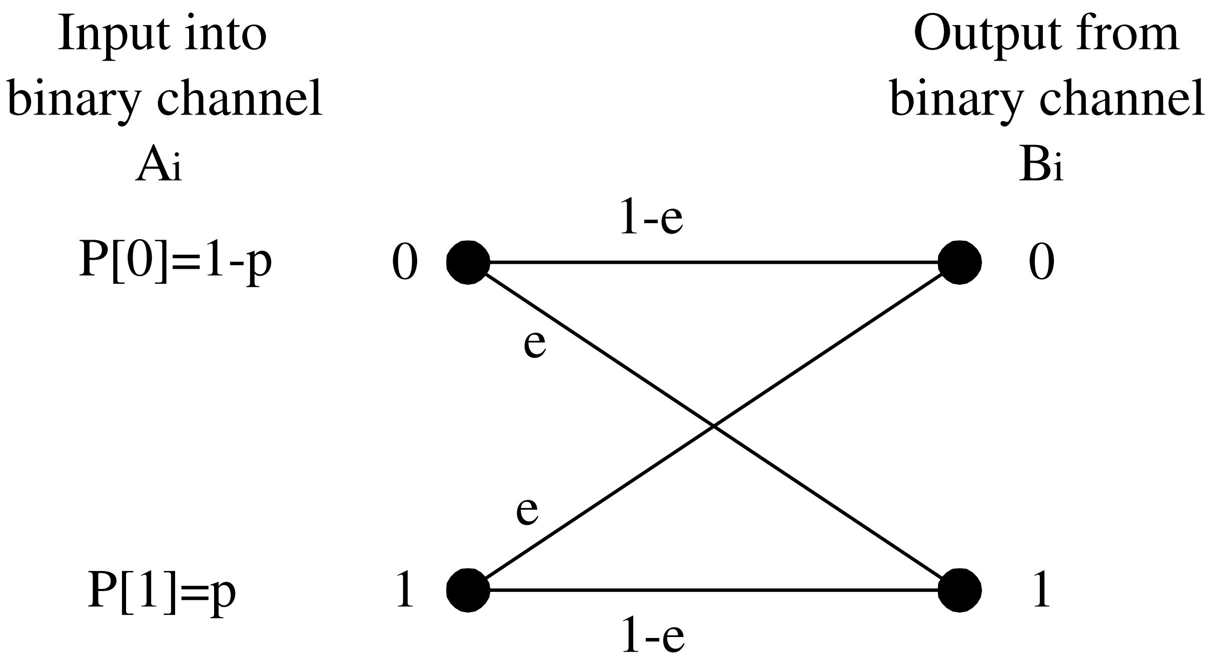

S
A
B∩A
B

Example: Two black balls and three white balls are in an urn. Two balls are selected at random without replacement. Find the probability that both balls are black.

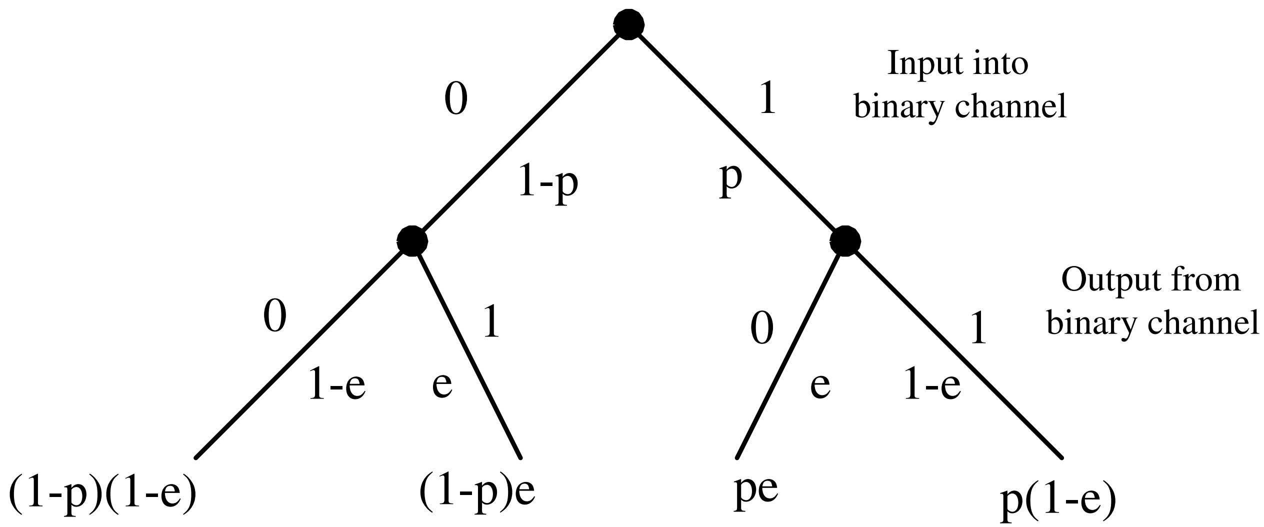


$P[B1 \cap B2] = P[B2|B1]P[B1] = (2/5)(1/4) = 1/10.$

Example: Communication Systems

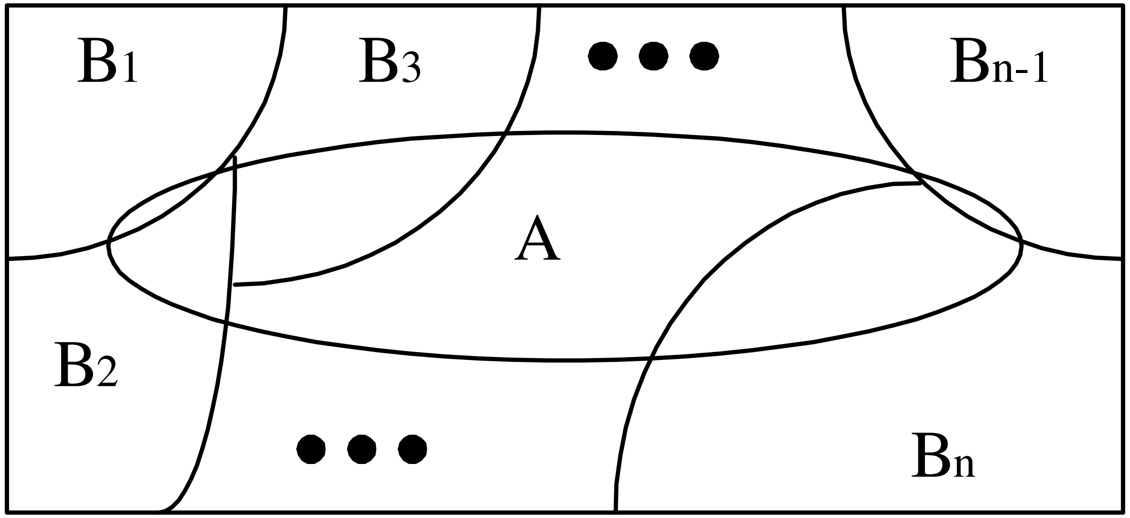


Find $P[A_i \cap B_j]$ for all $i, j = 0, 1$.



$P[A_0 \cap B_0] = (1-p)(1-e)$, $P[A_0 \cap B_1] = (1-p)e$, $P[A_1 \cap B_0] = pe$, and $P[A_1 \cap B_1] = p(1-e)$.

Let $B_1, B_2, \ldots, B_n$ be mutually exclusive events and $S = B_1 \cup B_2 \cup \cdots \cup B_n$



Any event A can be partitioned.

$$A = A \cap S$$

$$\begin{aligned} &= A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) \\ &= (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n) \end{aligned}$$

$$\begin{aligned} P[A] &= P[A \cap B_1] + P[A \cap B_2] + \cdots + P[A \cap B_n] \\ &= P[A|B_1]P[B_1] + P[A|B_2]P[B_2] + \\ &\quad \cdots + P[A|B_n]P[B_n] \end{aligned}$$

Bayes' Rule

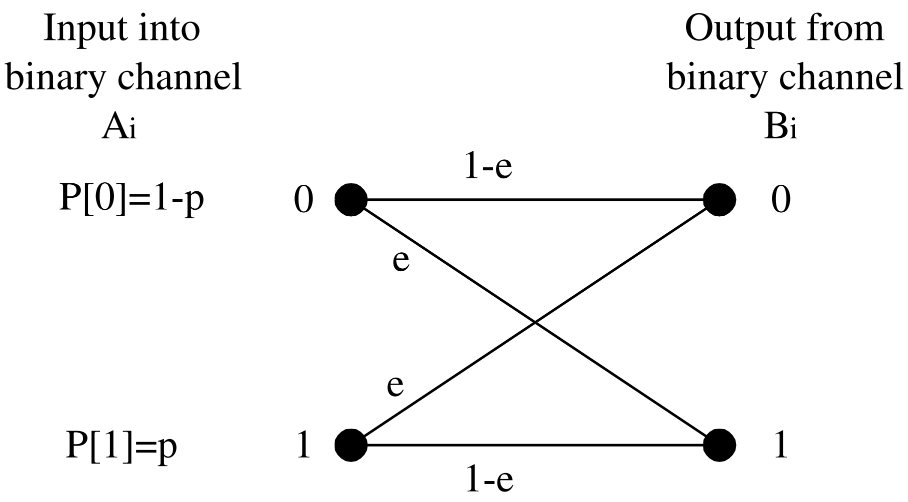
Let $B_1, B_2, \ldots, B_n$ be a partition of S. Suppose A occurs; what is the probability of event B_j?

$$P[B_j|A] = \frac{P[A \cap B_j]}{P[A]} = \frac{P[A|B_j]P[B_j]}{\sum_{k=1}^{n} P[A|B_k]P[B_k]}$$

$P[B_j]$: *a priori* probability

$P[B_j|A]$: *a posterior* probability

Example:



Assume $p = 1/2$.
Then

$$\begin{aligned} P[B_1] &= P[B_1|A_0]P[A_0] + P[B_1|A_1]P[A_1] \\ &= e(1/2) + (1-e)(1/2) = 1/2 \end{aligned}$$

$$
\begin{aligned}
P[A_0|B_1] &= \frac{P[B_1|A_0]P[A_0]}{P[B_1]} = \frac{e/2}{1/2} = e \\
P[A_1|B_1] &= \frac{P[B_1|A_1]P[A_1]}{P[B_1]} = \frac{(1-e)/2}{1/2} = 1-e
\end{aligned}
$$

When $e < 1/2$, $P[A_0|B_1] < P[A_1|B_1]$.

Independency of Events

- Define events A and B to be independent if $P[A \cap B] = P[A]P[B]$

$$\rightarrow \quad P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[A]P[B]}{P[B]} = P[A].$$

Similarly, $P[B|A] = P[B]$.

- If $A \cap B = \emptyset$ and $P[A] \neq 0$ and $P[B] \neq 0$, then A, B cannot be independent.

The events $A_1, A_2, \ldots, A_n$ are said to be independent if, for $k = 2, \ldots, n$,

$$P[A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}] = P[A_{i_1}]P[A_{i_2}] \cdots P[A_{i_k}]$$

$1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

Sequential Experiments

- Subexperiments: E_1, E_2, ..., E_n.
- Outcome: $s = (s_1, s_2, \ldots, s_n)$.
- Sample space $S = S_1 \times S_2 \times \cdots \times S_n$.
- Let $A_1, A_2, \ldots, A_n$ be events such that A_k concerns only the outcome of the kth subexperiment.

- Assume that subexperiments are independent. It is reasonable to assume that

$$P[A_1 \cap A_2 \cap \cdots \cap A_n] = P[A_1]P[A_2] \cdots P[A_n].$$

Binomial Probability Law

Bernoulli Trial: Given event A.

"SUCESS" if A occurs;

"FAILURE" otherwise.

Example: A coin is tossed three times. Assume the tosses are independent with $P[H] = p$.

$$\begin{aligned}
P[\{HHH\}] &= P[\{H\}]P[\{H\}]P[\{H\}] = p^3 \\
P[\{HHT\}] &= P[\{H\}]P[\{H\}]P[\{T\}] = p^2(1-p) \\
P[\{HTH\}] &= P[\{H\}]P[\{T\}]P[\{H\}] = p^2(1-p) \\
P[\{THH\}] &= P[\{T\}]P[\{H\}]P[\{H\}] = p^2(1-p) \\
P[\{TTH\}] &= P[\{T\}]P[\{T\}]P[\{H\}] = p(1-p)^2 \\
P[\{THT\}] &= P[\{T\}]P[\{H\}]P[\{T\}] = p(1-p)^2 \\
P[\{HTT\}] &= P[\{H\}]P[\{T\}]P[\{T\}] = p(1-p)^2
\end{aligned}$$

$$P[\{TTT\}] = P[\{T\}]P[\{T\}]P[\{T\}] = (1-p)^3$$

k: the number of heads in three trials

$$
\begin{aligned}
P[k=0] &= P[\{TTT\}] = (1-p)^3 \\
P[k=1] &= P[\{TTH\}] + P[\{THT\}] + P[\{HTT\}] \\
&= 3p(1-p)^2 \\
P[k=2] &= P[\{HHT\}] + P[\{HTH\}] \\
&+ P[\{THH\}] = 3p^2(1-p) \\
P[k=3] &= P[\{HHH\}] = p^3
\end{aligned}
$$

Theorem 1 *Let k be the number of successes in n independent Bernoulli trials, then the probabilities of k are given by the binomial probability law:*

$$p_n(k) = \binom{n}{k} p^k (1-p)^{n-k},$$

where p is the probability of success in a Bernoulli trial and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Geometric Probability Law

- Repeat Bernoulli trials until the first success occurs.
- $p(m)$: probability of m trials are required to success.
- A_i: success in the ith trial.
- Geometric probability law

$$\begin{aligned} p(m) &= P[A_1^c \cap A_2^c \cap \cdots \cap A_{m-1}^c \cap A_m] \\ &= (1-p)^{m-1}p \qquad m = 1, 2, \cdots \end{aligned}$$

- Verify

$$\sum_{m=1}^{\infty} p(m) = 1.$$

- Let $q = 1 - p$. Then

$$\begin{aligned} P[m > K] &= p \sum_{m=K+1}^{\infty} q^{m-1} = pq^K \sum_{j=0}^{\infty} q^j \\ &= pq^K \frac{1}{1-q} = q^K \end{aligned}$$