

PMDS Array Codes With Small Sub-packetization, Small Repair Bandwidth/Rebuilding Access

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Abstract—Partial maximum distance separable (PMDS) codes are a kind of erasure codes where the nodes are divided into multiple groups with each forming an MDS code with a smaller code length, thus they allow repairing a failed node with only a few helper nodes and can correct all erasure patterns that are information-theoretically correctable. However, the repair of a failed node of PMDS codes still requires a large amount of communication if the group size is large. Recently, PMDS array codes with each local code being an MSR code were introduced to reduce the repair bandwidth further. However, they require extensive rebuilding access and unavoidably a significant sub-packetization level. In this paper, we first propose two constructions of PMDS array codes with two global parities that have smaller sub-packetization levels and much smaller finite fields than the existing one. One construction can support an arbitrary number of local parities and has $(1 + \epsilon)$ -optimal repair bandwidth (i.e., $(1 + \epsilon)$ times the optimal repair bandwidth), while the other one is limited to two local parities but has significantly smaller rebuilding access and its sub-packetization level is only 2. In addition, we present a construction of PMDS array code with three global parities, which has a smaller sub-packetization level as well as $(1 + \epsilon)$ -optimal repair bandwidth, the required finite field is significantly smaller than existing ones.

Index Terms—Array codes, partial MDS codes, rebuilding access, repair bandwidth, sub-packetization.

I. INTRODUCTION

With the rapid increase in the volumes of data stored online, traditional storage techniques such as duplicating or triplicating data are not economically feasible. This has resulted in erasure coding-based distributed storage systems, which can provide reliability with low storage overhead. Previous distributed storage systems usually call upon the maximum distance separable (MDS) codes, which provide the optimal tradeoff between fault tolerance and storage overhead. However, the downside of employing MDS codes is the excessive repair bandwidth and rebuilding access when repairing a failed node, where *repair bandwidth* is defined as the amount of data downloaded from helper nodes to repair a failed node and

rebuilding access is defined as the amount of data accessed. Consider a distributed storage system that is based on an $[n, k]$ MDS code, repairing a failed node requires accessing and downloading the entire content from any k surviving nodes, thus leading to a large amount of access and bandwidth.

To reduce the repair bandwidth, regenerating codes were introduced in the pioneering work [2], which allow for repairing a failed node by contacting more than k surviving nodes but only downloading a fraction of the data stored at each node. The optimal tradeoff between the storage and repair bandwidth was also characterized in [2], which leads to two extremal classes of codes, namely *minimum storage regenerating (MSR)* codes and *minimum bandwidth regenerating (MBR)* codes. MSR codes are a kind of MDS codes as they are optimal in terms of storage overhead, whereas MBR codes result in more storage overhead but can offer the minimum repair bandwidth. Regenerating codes including the MSR codes and MBR codes have attracted a lot of attention in the past decade [3]–[22].

Although regenerating codes can significantly reduce the repair bandwidth, however, a large number of helper nodes are required to be contacted when repairing a failed node. As an alternative and parallel coding technique, locally repairable codes (LRCs) require only a few helper nodes during the repair process, however, at the cost of introducing additional redundancy to the system. Studies on the upper bound of the minimum distance of LRCs as well as the optimal constructions have also attracted a lot of attention in the past decade [23]–[32]. In another line of research, Partial MDS (PMDS) codes [33] provide an alternative solution, which are a strictly stronger class of LRCs as they are not only distance-optimal LRCs but can also correct any erasure pattern that is information-theoretically correctable. Formally, a $(\mu, n; r, s)$ PMDS code is a $[\mu n, \mu(n - r) - s]$ linear code, which can be partitioned into μ groups each of size n , such that any erasure pattern with r erasures in each group plus any s erasures in arbitrary places can be tolerated. Note that maximally recoverable codes are also referred to as PMDS codes [34], [35] when restricted to the RAID-type architecture. Besides, they can be applied to more topologies, e.g., see [36] for grid-like topologies and [37] for product topologies.

In general, PMDS codes are much harder to obtain than LRC codes, especially over relatively small finite fields. Existing results [33], [35], [39]–[44] show that a finite field with an exponential size is required when $r \geq 2$ and the number s of global parities exceeds 3. Note that repairing a failed node of PMDS codes may still require a large amount of communication if the group size is large, as the whole content of the

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TABLE I

COMPARISON OF KEY PARAMETERS AMONG NEW $(\mu, n; r, s = 2)$ PMDS ARRAY CODES AND THE ONE IN [38], WHERE $n' \geq 2$ AND WE ASSUME $n' | n$ FOR CONVENIENCE OF NOTATION

	r	s	Sub-packetization level ℓ	Field size q	Repair bandwidth γ	Rebuilding access Γ
Construction A in [38]	≥ 2	2	r^n	$> \mu r(rn - r + n - 2)$	$\frac{\ell}{r}(n - 1)$	$\ell(n - 1)$
Construction 2 (Thm. 1)	≥ 2	2	$r^{n'}$	$> \mu r n' \lceil \frac{n}{r n'} \rceil$	$(1 + \frac{(\frac{n'}{r} - 1)(r - 1)}{n - 1}) \frac{\ell}{r}(n - 1)$	$\ell(n - 1)$
Construction 3 (Thms. 2 and 3)	2	2	2	$> \mu n$	$\frac{3n}{2} - 2$	$\frac{3n}{2} - 2$

TABLE II

COMPARISON OF KEY PARAMETERS AMONG NEW $(\mu, n; r, s = 3)$ PMDS ARRAY CODES AND EXISTING ONES IN [38], WHERE WE ASSUME $n' | n$ FOR CONVENIENCE OF NOTATION

	r	s	Sub-packetization ℓ	Field size q	Repair bandwidth γ	Rebuilding access Γ
Construction B in [38]	≥ 2	3	r^n	$\geq (rn)^{\mu(n-r)}$	$\frac{\ell}{r}(n - 1)$	$\ell(n - 1)$
Construction C in [38]	≥ 2	3	r^n	$\geq \max\{rn, \mu + 1\}^{n-r}$	$\frac{\ell}{r}(n - 1)$	$\ell(n - 1)$
Construction D in [38]	≥ 2	3	r^n	$\geq rn(\mu n)^{3(r+1)-1}$	$\frac{\ell}{r}(n - 1)$	$\ell(n - 1)$
Construction 4 (Thm. 4)	≥ 2	3	$r^{n'}$	$> (\mu r n' \lceil \frac{n}{r n'} \rceil + 1)^3$	$(1 + \frac{(\frac{n'}{r} - 1)(r - 1)}{n - 1}) \frac{\ell}{r}(n - 1)$	$\ell(n - 1)$

surviving nodes in the same group needs to be downloaded. LRCs also have the same issue. In [24], [25], the idea of using regenerating codes (including MSR codes and MBR codes) to encode the local group was first introduced to LRCs for reducing the repair bandwidth further. Very recently, by using MSR codes to encode the local group, PMDS array codes were introduced in [38], which combines both the advantages of PMDS codes and MSR codes, i.e., 1) Can correct any erasure pattern that is information-theoretically correctable, 2) Require fewer helper nodes during the node repair process, and 3) Can further reduce the repair bandwidth. However, the explicit PMDS code constructions with a linear field size were only provided for two global parities, the constructions of PMDS array codes for more than two global parities still require a relatively large finite field. Although the PMDS array codes in [38] have the optimal repair bandwidth as each local group is encoded by an MSR code, they require large rebuilding access and unavoidably a large sub-packetization level.

In this paper, we focus on PMDS array codes with smaller sub-packetization levels and smaller field sizes than existing ones while endowing $(1 + \epsilon)$ -optimal repair bandwidth. More specifically, we restrict to PMDS array codes with two and three global parities, as it is promising to construct PMDS array codes with few global parities over small finite fields. We present two PMDS array codes with $s = 2$ global parities and one with $s = 3$ global parities. Comparisons of the key parameters among the newly proposed PMDS array codes and some existing ones are given in Tables I and II under $s = 2$ and $s = 3$, respectively.

From Tables I and II, we see that the new proposed PMDS array codes have the following advantages:

- The required finite fields of the two new PMDS array codes in Constructions 2 and 3 are only a fraction of around $\frac{1}{r}$ to $\frac{1}{r^2}$ as Construction A in [38], which is based on the first MSR code construction in [12] (i.e., each sub-stripe can be viewed as a scalar MDS code and one stripe contains ℓ sub-stripes) and the PMDS codes in [39].
- The new PMDS array code in Construction 3, which supports only two local parity nodes, has smaller re-

building access (when normalized by the file size) and a smaller sub-packetization level compared with the code in Construction A in [38] and the new code in Construction 2.

- The required field size of the new code in Construction 4 is significantly smaller than those in Constructions B-D in [38] for almost all parameter ranges, where Constructions B, C, and D in [38] are obtained by combining a universal PMDS code and an MSR code with each sub-stripe being a scalar MDS code (e.g., the first MSR code construction in [12]). More specifically, they employ the Gabidulin-code-based PMDS code in [25], the linearized-RS-codes-based PMDS code in [45], and a generalization of the PMDS code in [41] as the universal PMDS code, respectively.
- The new codes in Constructions 2 and 4 can provide a flexible tradeoff between the sub-packetization level and the repair bandwidth by varying n' in $[r + 1, n + 1)$ when the MDS array code in [46] with this flexible tradeoff is employed as the local code. The two extreme points of the tradeoff are $(\ell, \gamma) = (r^{r+1}, (1 + \frac{(\frac{n'}{r+1} - 1)(r - 1)}{n - 1}) \frac{\ell}{r}(n - 1))$ and $(\ell, \gamma) = (r^n, \frac{\ell}{r}(n - 1))$, where the latter one is also achieved by the constructions in [38]. As an example, when $n = 30$ and $r = 2$, some of the sub-packetization levels and repair bandwidths that the new codes in Constructions 2 and 4 can provide are

$$(\ell, \gamma) = \left(2^3, (1 + \frac{9}{29})\gamma^*\right), \left(2^5, (1 + \frac{5}{29})\gamma^*\right), \\ \left(2^6, (1 + \frac{4}{29})\gamma^*\right), \left(2^{10}, (1 + \frac{2}{29})\gamma^*\right), \\ (2^{30}, \gamma^*),$$

where all points except for the last one are new that can be achieved by Constructions 2 and 4, and $\gamma^* = \frac{\ell}{r}(n - 1)$ denotes the repair bandwidth of an $[n, n - r]$ MSR code with sub-packetization level ℓ .

- When $n = n'$, i.e., each local group of the new Constructions 2 and 4 forms an MSR code, the sub-packetization level ℓ and repair bandwidth γ are the same as those of Constructions B-D in [38]. Nevertheless, the required field sizes of the new code Constructions 2 and 4 are still smaller than the PMDS array code Construction A in [38] and Constructions B-D in [38], respectively, while all the other properties are the same.

Although Constructions B-D in [38] require a huge finite field, it is worth pointing out that they work for all $s \geq 1$.

The rest of this paper is organized as follows. Section II introduces some necessary preliminaries. Section III proposes two new PMDS array code constructions with two global parities and $(1 + \epsilon)$ -optimal repair bandwidth. A new PMDS array code construction with three global parities and $(1 + \epsilon)$ -optimal repair bandwidth is presented in Section IV. Finally, Section V draws the conclusion.

II. PRELIMINARIES

First of all, we fix some notations used in this paper. Let q be a prime power and \mathbf{F}_q the finite field containing q elements. For two integers a and b with $a < b$, denote by $[a, b)$ the set $\{a, a + 1, \dots, b - 1\}$.

Let $\ell \geq 1$, for an $al \times bl$ matrix A , let $A(j)$ denote the j -th column of A , and $A(J)$ denote the sub-matrix of A that formed by the columns of A with indices in the set J where $J \subset [0, bl)$. If $J = [j\ell, j\ell + 1, \dots, j\ell + \ell)$ for some $j \in [0, b)$, then we say that the sub-matrix $A(J)$ is the j -th *thick* column of A and denote it by $A[j]$. Let $A[J]$ denote the sub-matrix of A formed by the thick columns of A with indices indicated by $J \subset [0, b)$. Similarly, let $A^{[j]}$ and $A^{[J]}$ denote the j -th thick row of A and the sub-matrix formed by the thick rows of A with indices in the set J , respectively, where $j \in [0, a)$ and $J \subset [0, a)$. For ℓ matrices $B_0, B_1, \dots, B_{\ell-1}$, $\text{blkdiag}(B_0, B_1, \dots, B_{\ell-1})$ denotes the block diagonal matrix

$$\begin{pmatrix} B_0 & & & \\ & B_1 & & \\ & & \ddots & \\ & & & B_{\ell-1} \end{pmatrix}.$$

A. MDS Array Codes

An $[n, n - r, \ell]$ linear array code \mathcal{C} over \mathbf{F}_q has $n - r$ information nodes and r parities nodes in each codeword, with each node (or codeword symbol) being a column vector of length ℓ over \mathbf{F}_q , where ℓ is referred to as the *sub-packetization level*. The following lemma can be utilized to verify whether an array code is MDS.

Lemma 1. ([12]) For an $[n, n - r, \ell]$ array code over \mathbf{F}_q admitting the following parity-check matrix

$$H = \begin{pmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,n-1} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r-1,0} & A_{r-1,1} & \cdots & A_{r-1,n-1} \end{pmatrix},$$

where $A_{i,j}$ is an $\ell \times \ell$ matrix over \mathbf{F}_q , it is an MDS array code if and only if any $r \times r$ sub-block matrix $H[J]$ of H is nonsingular, where $J \subset [0, n)$ and $|J| = r$.

In [2], the repair bandwidth $\gamma(d)$ of $[n, n - r, \ell]$ MDS array codes has shown to be

$$\gamma(d) \geq \frac{d\ell}{d - (n - r) + 1},$$

where $d \in [n - r, n)$ is the number of contacted helper nodes. MDS array codes with the repair bandwidth attaining the above lower bound are said to have the *optimal repair bandwidth*, and are exactly MSR codes. If the repair bandwidth of an MDS array code is $(1 + \epsilon)$ times the above lower bound where $\epsilon < 1$ is a small constant, we say that the MDS array code has $(1 + \epsilon)$ -*optimal repair bandwidth*, which also referred to as *near-optimal repair bandwidth* in [47].

Subsequently, the rebuilding access $\Gamma(d)$ of $[n, n - r, \ell]$ MDS array codes has shown to be

$$\Gamma(d) \geq \frac{d\ell}{d - (n - r) + 1}.$$

MDS array codes with the rebuilding access attaining the above lower bound are said to have the *optimal rebuilding access*. Clearly, MDS array codes with optimal rebuilding access will also have optimal repair bandwidth, but not vice versa. In the literature, most of the MDS array code constructions focus on $d = n - 1$ so as to maximally reduce the repair bandwidth, this is also the setting of this work unless otherwise stated.

B. PMDS Array Codes

Now we give the formal definition of PMDS array codes.

Definition 1. ([38]) Let \mathcal{C} be a $[\mu n, \mu(n - r) - s, \ell]$ linear code over \mathbf{F}_q , it is said to be a $(\mu, n; r, s)$ PMDS array code with sub-packetization level ℓ if

- For any $i \in [0, \mu)$, restricting \mathcal{C} to coordinates in $W_i = [ni, ni + n)$ yields an $[n, n - r, \ell]$ MDS array code over \mathbf{F}_q , where $\mathcal{C}|_{W_i}$ is usually referred to as the i -th local code and W_i the i -th local group.
- For any $E_i \subset W_i$ with $|E_i| = r$, removing the coordinates of \mathcal{C} in $\bigcup_{i=0}^{\mu-1} E_i$ yields a $[\mu(n - r), \mu(n - r) - s, \ell]$ MDS array code over \mathbf{F}_q , i.e., the code \mathcal{C} can correct up to r erasures in W_i for $i \in [0, \mu)$ plus s erasures anywhere.

Particularly, PMDS array codes over \mathbf{F}_q will be referred to as *PMDS codes* if $\ell = 1$.

From Definition 1, it is immediate that every $(\mu, n; r, s)$ PMDS array code with sub-packetization ℓ permits a form of the following parity-check matrix

$$\mathcal{H} = \begin{pmatrix} H_0 & & & \\ & H_1 & & \\ & & \ddots & \\ & & & H_{\mu-1} \\ P_0 & P_1 & \cdots & P_{\mu-1} \end{pmatrix}, \quad (1)$$

where H_i is an $r\ell \times n\ell$ matrix and P_i is an $s\ell \times n\ell$ matrix, μ, n, r and s denote the number of local groups, the size of each local group (or the code length of each local code), the

number of local parities in each local code, and the number of global parities, respectively.

In the following, we introduce several lemmas, which are very useful when checking i) and ii) of Definition 1. The immediately following lemma helps to check i) of Definition 1, i.e., each local code is MDS.

Lemma 2. (Block Vandermonde matrix, [12, Lemma 14]) Let B_0, \dots, B_{r-1} be $\ell \times \ell$ matrices such that $B_i B_j = B_j B_i$ and $B_i - B_j$ is nonsingular for all $i, j \in [0, r)$ with $i \neq j$, then the matrix

$$\begin{pmatrix} I & I & \cdots & I \\ B_0 & B_1 & \cdots & B_{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ B_0^{r-1} & B_1^{r-1} & \cdots & B_{r-1}^{r-1} \end{pmatrix}$$

is nonsingular.

The following two lemmas help to verify the non-singularity of some key matrices when checking whether the array codes proposed in Section III-A and Section IV satisfy ii) of Definition 1, respectively.

Lemma 3. ([42, Lemma 2]) Let C_0, \dots, C_{s-1} be $r \times (r+1)$ matrices and D_0, \dots, D_{s-1} be $s \times (r+1)$ matrices, and let $D_i^{(j)}$ be the j -th row of D_i . Then

$$\begin{aligned} & (-1)^{\frac{rs(s-1)}{2}} \det \begin{pmatrix} C_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & C_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & C_{s-1} \\ D_0 & D_1 & \cdots & D_{s-1} \end{pmatrix} \\ &= \det \begin{pmatrix} \det \begin{pmatrix} C_0 \\ D_0^{(0)} \end{pmatrix} & \cdots & \det \begin{pmatrix} C_{s-1} \\ D_{s-1}^{(0)} \end{pmatrix} \\ \vdots & \ddots & \vdots \\ \det \begin{pmatrix} C_0 \\ D_0^{(s-1)} \end{pmatrix} & \cdots & \det \begin{pmatrix} C_{s-1} \\ D_{s-1}^{(s-1)} \end{pmatrix} \end{pmatrix}. \end{aligned}$$

Lemma 4. ([42, Lemma 5]) Let C_0 be an $r \times (r+1)$ matrix, C_1 be an $r \times (r+2)$ matrix, D_0 be a $3 \times (r+1)$ matrix and D_1 be a $3 \times (r+2)$ matrix, and let $D_i^{(j)}$ be the j -th row of D_i . Then

$$\begin{aligned} & \det \begin{pmatrix} C_0 & 0 \\ 0 & C_1 \\ D_0 & D_1 \end{pmatrix} = 0 \iff \det \begin{pmatrix} C_0 \\ D_0^{(0)} \end{pmatrix} \det \begin{pmatrix} C_1 \\ D_1^{(1)} \\ D_1^{(2)} \end{pmatrix} \\ & - \det \begin{pmatrix} C_0 \\ D_0^{(1)} \end{pmatrix} \det \begin{pmatrix} C_1 \\ D_1^{(0)} \\ D_1^{(2)} \end{pmatrix} + \det \begin{pmatrix} C_0 \\ D_0^{(2)} \end{pmatrix} \det \begin{pmatrix} C_1 \\ D_1^{(0)} \\ D_1^{(1)} \end{pmatrix} = 0. \end{aligned}$$

The following lemma helps to calculate some key determinants involved in the RHS of the formulas in Lemmas 3 and 4 when checking whether the array code proposed in Section IV satisfies ii) of Definition 1.

Lemma 5. (Cauchy–Vandermonde matrix, [48, Proposition 4.1]) Let

$$V = \begin{pmatrix} \frac{1}{c_0-d_0} & \cdots & \frac{1}{c_0-d_{l-1}} & 1 & c_0 & \cdots & c_0^{n-l-1} \\ \frac{1}{c_1-d_0} & \cdots & \frac{1}{c_1-d_{l-1}} & 1 & c_1 & \cdots & c_1^{n-l-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{c_{n-1}-d_0} & \cdots & \frac{1}{c_{n-1}-d_{l-1}} & 1 & c_{n-1} & \cdots & c_{n-1}^{n-l-1} \end{pmatrix}$$

be a Cauchy–Vandermonde matrix, then

$$\det(V) = \frac{\left(\prod_{0 \leq i < j < n} (c_j - c_i) \right) \left(\prod_{0 \leq i < j < l} (d_i - d_j) \right)}{\prod_{0 \leq i < n, 0 \leq j < l} (c_i - d_j)}.$$

C. Partition of Basis $\{e_0, \dots, e_{N-1}\}$

In this subsection, we revisit a series of particular partitions of a basis set that was proposed in [11] and [46], which will facilitate the understanding of the new constructions in this paper.

For any two integers $r, m \geq 2$, let e_0, \dots, e_{r^m-1} be a basis of $\mathbf{F}_q^{r^m}$. For example, they can be simply set as the standard basis, i.e.,

$$e_i = (0, \dots, 0, 1, 0, \dots, 0), \quad i \in [0, r^m),$$

where only the i -th entry is 1.

For consistency, we follow the notation in [11], [46]. Given an integer $0 \leq a < r^m$, denote by (a_0, \dots, a_{m-1}) its r -ary expansion, i.e., $a = \sum_{j=0}^{m-1} r^{m-1-j} a_j$. For $0 \leq i < m$ and $0 \leq t < r$, define a subset of $\{e_0, \dots, e_{r^m-1}\}$ as

$$V_{i,t} = \{e_a | a_i = t, 0 \leq a < r^m\}, \quad (2)$$

where a_i is the i -th element in the r -ary expansion of a .

Straightforwardly, $|V_{i,t}| = r^{m-1}$, and $\{V_{i,0}, V_{i,1}, \dots, V_{i,r-1}\}$ is a partition of the set $\{e_0, \dots, e_{r^m-1}\}$ for any $i \in [0, m)$. Table III gives two examples of the set partitions defined in (2).

TABLE III

(A) AND (B) DENOTE THE m PARTITIONS OF THE SET $\{e_0, \dots, e_{r^m-1}\}$ DEFINED BY (2) FOR $m = 3, r = 2$, AND $m = 2, r = 3$, RESPECTIVELY.

i	0	1	2	i	0	1	2
$V_{i,0}$	e_0	e_0	e_0	$V_{i,1}$	e_4	e_2	e_1
	e_1	e_1	e_2		e_5	e_3	e_3
	e_2	e_4	e_4		e_6	e_6	e_5
	e_3	e_5	e_6		e_7	e_7	e_7

(A)

i	0	1	i	0	1	i	0	1
$V_{i,0}$	e_0	e_0	$V_{i,1}$	e_3	e_1	$V_{i,2}$	e_6	e_2
	e_1	e_3		e_4	e_4		e_7	e_5
	e_2	e_6		e_5	e_7		e_8	e_8

(B)

For the convenience of notation, we also denote by $V_{i,t}$ the $r^{m-1} \times r^m$ matrix whose rows are formed by vectors e_i in their corresponding sets, such that i is sorted in ascending order. For example, when $r = 2$ and $m = 3$, $V_{1,0}$ can be viewed as a 4×8 matrix as follows

$$V_{1,0} = (e_0^\top \ e_1^\top \ e_4^\top \ e_5^\top)^\top,$$

where \top represents the transpose operator.

D. Review of an $[n, n - r, \ell]$ MDS Array Code in [46] With Small Sub-packetization Level

Construction 1. (The code \mathcal{C}_5 in [46]) Let r, n', n be three positive integers, where $r \geq 2$ and $n \geq n'$. For $i \in [0, n)$, denote by \bar{i} the integer in $[0, n')$ such that $i \equiv \bar{i} \pmod{n'}$, i.e., $\bar{i} = i \% n'$ for short, with $\%$ denoting the modulo operation. Let $\ell = r^n$, then an $[n, n - r, \ell]$ array code over \mathbf{F}_q is defined by the following parity-check matrix

$$H = \begin{pmatrix} I & I & \cdots & I \\ A_0 & A_1 & \cdots & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_0^{r-1} & A_1^{r-1} & \cdots & A_{n-1}^{r-1} \end{pmatrix}, \quad (3)$$

where $A_i, i \in [0, n)$ satisfy

$$\begin{pmatrix} V_{i,0} \\ V_{i,1} \\ \vdots \\ V_{i,r-1} \end{pmatrix} A_i = \begin{pmatrix} \lambda_{i,0} V_{i,0} \\ \lambda_{i,1} V_{i,1} \\ \vdots \\ \lambda_{i,r-1} V_{i,r-1} \end{pmatrix}, \quad (4)$$

with $\lambda_{i,t} \in \mathbf{F}_q \setminus \{0\}$ and $V_{i,t}$ being defined by (2) for $t \in [0, r)$.

Clearly from (2) and (4), we have that A_i is a diagonal matrix and the a -th row of A_i is

$$e_a A_i = \lambda_{i, a_{\bar{i}}} e_a, \quad (5)$$

where $i \in [0, n)$, $a \in [0, \ell)$, and $a_{\bar{i}}$ denotes the \bar{i} -th element in the r -ary expansion of a .

In the following, we revisit the results related to the code \mathcal{C}_5 in [46].

Lemma 6. (Theorems 13 and 14 of [46]) The array code in Construction 1 is MDS with the repair bandwidth γ_i of node i ($i \in [0, n)$) being

$$\gamma_i = \begin{cases} (1 + \frac{(\lceil \frac{n}{n'} \rceil - 1)(r-1)}{n-1}) \frac{\ell}{r} (n-1), & \text{if } 0 \leq i \% n' < n \% n', \\ (1 + \frac{(\lceil \frac{n}{n'} \rceil - 1)(r-1)}{n-1}) \frac{\ell}{r} (n-1), & \text{otherwise,} \end{cases} \quad (6)$$

if the following requirements can be satisfied

- R1. $\lambda_{i,u} \neq \lambda_{j,v}$ for all $u, v \in [0, r)$ and $i, j \in [0, n)$ with $j \not\equiv i \pmod{n'}$,
- R2. $\lambda_{i,u} \neq \lambda_{i+gn',u}$ for all $u \in [0, r)$, $g \in [1, \lceil \frac{n}{n'} \rceil)$, $i \in [0, n')$ with $i + gn' < n$,
- R3. $\lambda_{i,0}, \lambda_{i,1}, \dots, \lambda_{i,r-1}$ are pairwise distinct for every $i \in [0, n)$.

The rebuilding access Γ_i of node i is

$$\Gamma_i = \ell(n-1) \text{ for } i \in [0, n)$$

when the repair bandwidth achieves the value in (6).

Lemma 7. (Theorem 15 of [46]) The three requirements R1-R3 in Lemma 6 can be satisfied if the finite field \mathbf{F}_q contains at least Φ nonzero elements, where

$$\Phi = \begin{cases} rn'(\lceil \frac{n}{rn'} \rceil - 1) + (n \% n')r, & \text{if } 0 < n \% (rn') < n', \\ rn' \lceil \frac{n}{rn'} \rceil, & \text{otherwise.} \end{cases} \quad (7)$$

Furthermore, $A_i A_j = A_j A_i$ and $A_i - A_j$ is nonsingular for all $i, j \in [0, n)$ with $i \neq j$.

III. NEW PMDS ARRAY CODE CONSTRUCTIONS WITH TWO GLOBAL PARITIES

In this section, we present two PMDS array code constructions with two global parities. The first one allows an arbitrary number r of local parities, and has $(1 + \epsilon)$ -optimal repair bandwidth but high rebuilding access, while the second one has both $(1 + \epsilon)$ -optimal repair bandwidth and smaller rebuilding access when compared with the first one, but can only support two local parities, i.e., $r = 2$.

A. A New $(\mu, n; r, s = 2)$ PMDS Array Code Construction

Construction 2. Let μ, r, n', n be four positive integers, where $\mu, r \geq 2$ and $n > n'$, and let $\ell = r^n$. We construct a new $[\mu n, \mu(n-r) - 2, \ell]$ array code over \mathbf{F}_q with the parity-check matrix having the form as in (1), where

$$H_i = H, \quad (8)$$

and

$$P_i = \begin{pmatrix} A_0^r & A_1^r & \cdots & A_{n-1}^r \\ \theta_i A_0^{r-1} & \theta_i A_1^{r-1} & \cdots & \theta_i A_{n-1}^{r-1} \end{pmatrix} \quad (9)$$

for $i \in [0, \mu)$, where H is defined in (3), i.e., the parity-check matrix of the code in Construction 1, A_i is an $\ell \times \ell$ matrix defined in (4), $\theta_i \in \mathbf{F}_q \setminus \{0\}$ for $i \in [0, \mu)$.

Remark 1. Note that in both Construction 2 above and Construction A in [38], a scalar multiplier is employed to distinguish different global parity-check blocks $P_0, P_1, \dots, P_{\mu-1}$ in (1).

Precisely, in (9), we employ independent scalar multipliers $\theta_0, \theta_1, \dots, \theta_{\mu-1}$ in $P_0, P_1, \dots, P_{\mu-1}$, which is motivated by the constructions in [42]. In addition, the sub-block matrix formed by any $r + 2$ thick columns of $\begin{pmatrix} H_i \\ P_i \end{pmatrix}$ or any $r + 1$

thick columns of $\begin{pmatrix} H_i \\ P_i^{[t]} \end{pmatrix}$ ($t = 0, 1$) is equivalent to a block Vandermonde matrix after row permutation and scaling. Thus the determinants of the sub-block matrices mentioned above can be easily calculated, which greatly facilitates the proof that Construction 2 gives a PMDS code.

Whereas in Construction A in [38], the multipliers are $\beta^{-N}, \beta^{-2N}, \dots, \beta^{-\mu N}$, where N is an integer and β has order at least μN in \mathbf{F}_q . It was proved in [38] that N should be larger than a threshold to guarantee the code is PMDS, which leads to a larger finite field than ours.

Theorem 1. Let $q > \mu\Phi$ be a prime power such that there exists a multiplicative subgroup G of $\mathbf{F}_q \setminus \{0\}$ of size at least Φ and with at least μ cosets, where Φ is defined in (7). Choosing $\lambda_{i,t}, i \in [0, n), t \in [0, r)$ from G , then the code in Construction 2 is a PMDS array code if $\theta_0, \dots, \theta_{\mu-1}$ are elements from distinct cosets of G , where the repair bandwidth and the rebuilding access of node $in + j$ are

$$\gamma_{in+j} = \begin{cases} (1 + \frac{(\lceil \frac{n}{n'} \rceil - 1)(r-1)}{n-1}) \frac{\ell}{r} (n-1), & \text{if } 0 \leq j \% n' < n \% n', \\ (1 + \frac{(\lceil \frac{n}{n'} \rceil - 1)(r-1)}{n-1}) \frac{\ell}{r} (n-1), & \text{otherwise,} \end{cases}$$

and

$$\Gamma_{in+j} = \ell(n-1)$$

$$(-1)^r \det(B_a) = \det \left(\begin{array}{c} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{j_0, a_{\bar{j}_0}} & \lambda_{j_1, a_{\bar{j}_1}} & \cdots & \lambda_{j_r, a_{\bar{j}_r}} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j_0, a_{\bar{j}_0}}^r & \lambda_{j_1, a_{\bar{j}_1}}^r & \cdots & \lambda_{j_r, a_{\bar{j}_r}}^r \end{pmatrix} \\ \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{j_0, a_{\bar{j}_0}} & \lambda_{j_1, a_{\bar{j}_1}} & \cdots & \lambda_{j_r, a_{\bar{j}_r}} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j_0, a_{\bar{j}_0}}^{r-1} & \lambda_{j_1, a_{\bar{j}_1}}^{r-1} & \cdots & \lambda_{j_r, a_{\bar{j}_r}}^{r-1} \\ \theta_i \lambda_{j_0, a_{\bar{j}_0}}^{-1} & \theta_i \lambda_{j_1, a_{\bar{j}_1}}^{-1} & \cdots & \theta_i \lambda_{j_r, a_{\bar{j}_r}}^{-1} \end{pmatrix} \\ \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{t_0, a_{\bar{t}_0}} & \lambda_{t_1, a_{\bar{t}_1}} & \cdots & \lambda_{t_r, a_{\bar{t}_r}} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{t_0, a_{\bar{t}_0}}^r & \lambda_{t_1, a_{\bar{t}_1}}^r & \cdots & \lambda_{t_r, a_{\bar{t}_r}}^r \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{t_0, a_{\bar{t}_0}}^{r-1} & \lambda_{t_1, a_{\bar{t}_1}}^{r-1} & \cdots & \lambda_{t_r, a_{\bar{t}_r}}^{r-1} \\ \theta_k \lambda_{t_0, a_{\bar{t}_0}}^{-1} & \theta_k \lambda_{t_1, a_{\bar{t}_1}}^{-1} & \cdots & \theta_k \lambda_{t_r, a_{\bar{t}_r}}^{-1} \end{pmatrix} \end{array} \right). \quad (11)$$

B. A New $(\mu, n; r = 2, s = 2)$ PMDS Array Code Construction With Small Rebuilding Access

Partly motivated by the construction of MSR codes with optimal rebuilding access in [22], we present the second PMDS array code construction, which deploys both triangular matrices and diagonal matrices as building blocks. This is the first time to use non-diagonal matrices as building blocks in PMDS array codes besides diagonal matrices, and thus leads to smaller rebuilding access (when normalized by the file size) and a smaller sub-packetization level, although it only supports two local parities.

Construction 3. For convenience of notation, we assume that n is even. Construct a $[\mu n, \mu(n-2) - 2, \ell = 2]$ linear array code over \mathbf{F}_q with the parity-check matrix having the form as in (1), where

$$H_i = H = \begin{pmatrix} I & I & \cdots & I \\ A_0 & A_1 & \cdots & A_{n-1} \end{pmatrix} \quad (12)$$

and

$$P_i = \begin{pmatrix} \lambda_0^2 I & \lambda_1^2 I & \cdots & \lambda_{n-1}^2 I \\ \theta_i \lambda_0^{-1} I & \theta_i \lambda_1^{-1} I & \cdots & \theta_i \lambda_{n-1}^{-1} I \end{pmatrix} \quad (13)$$

for $i \in [0, \mu)$, where I denotes the identity matrix of order 2,

$$A_i = \begin{cases} \begin{pmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{pmatrix}, & \text{if } 2 \mid i, \\ \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{pmatrix}, & \text{otherwise,} \end{cases} \quad (14)$$

and $\theta_0, \dots, \theta_{\mu-1}$ and $\lambda_0, \dots, \lambda_{n-1}$ are pairwise distinct nonzero elements in \mathbf{F}_q , respectively.

Theorem 2. Both the repair bandwidth and the rebuilding access of the array code in Construction 3 are $\frac{3n}{2} - 2$.

Proof. For a given b , where $b \in [0, \mu)$, let $f_0, \dots, f_{n-1} \in \mathbf{F}_q^2$ denote the data stored in the n nodes of the b -th local group. Then the b -th local code is subject to the following parity-check equations.

$$f_0 + f_1 + \cdots + f_{n-1} = \mathbf{0}, \quad (15)$$

$$A_0 f_0 + A_1 f_1 + \cdots + A_{n-1} f_{n-1} = \mathbf{0}. \quad (16)$$

Suppose node i of the b -th local group is failed, then we can repair it by contacting all the remaining nodes in the same local

group. In the following, we analyze the repair bandwidth and the rebuilding access.

- If $2 \mid i$, then let $e_0 = \begin{pmatrix} 1 & 0 \end{pmatrix}$. By multiplying e_0 with (15) and (16) from the left, we get

$$\begin{pmatrix} e_0 \\ e_0 A_i \end{pmatrix} f_i = - \sum_{j=0, j \neq i}^{n-1} \begin{pmatrix} e_0 \\ e_0 A_j \end{pmatrix} f_j.$$

Note that

$$\text{rank} \begin{pmatrix} e_0 \\ e_0 A_i \end{pmatrix} = \text{rank} \begin{pmatrix} 1 & 0 \\ \lambda_i & 1 \end{pmatrix} = 2,$$

and

$$\text{rank} \begin{pmatrix} e_0 \\ e_0 A_j \end{pmatrix} = \begin{cases} \text{rank} \begin{pmatrix} 1 & 0 \\ \lambda_j & 1 \end{pmatrix} = 2, & \text{if } 2 \mid j, \\ \text{rank} \begin{pmatrix} 1 & 0 \\ \lambda_j & 0 \end{pmatrix} = 1, & \text{otherwise.} \end{cases}$$

Thus the repair bandwidth γ_i of node i is

$$\gamma_i = \sum_{j=0, j \neq i}^{n-1} \text{rank} \begin{pmatrix} e_0 \\ e_0 A_j \end{pmatrix} = \left(\frac{n}{2} - 1\right) \times 2 + \frac{n}{2} = \frac{3n}{2} - 2,$$

and the rebuilding access Γ_i of node i is

$$\Gamma_i = \sum_{j=0, j \neq i}^{n-1} N_c \begin{pmatrix} e_0 \\ e_0 A_j \end{pmatrix} = \left(\frac{n}{2} - 1\right) \times 2 + \frac{n}{2} = \frac{3n}{2} - 2,$$

where $N_c(A)$ denotes the number of nonzero columns of the matrix A .

- If $2 \nmid i$, then let $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $S' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. By $S \times (15) + S' \times (16)$, we obtain

$$(S + S' A_i) f_i = - \sum_{j=0, j \neq i}^{n-1} (S + S' A_j) f_j.$$

Note that

$$\begin{aligned} S + S' A_i &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \lambda_i & 1 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}
 & S + S'A_j \\
 &= \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_j & 0 \\ 0 & \lambda_j \end{pmatrix}, & \text{if } 2 \nmid j, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_j & 1 \\ 0 & \lambda_j \end{pmatrix}, & \text{otherwise,} \end{cases} \\
 &= \begin{cases} \begin{pmatrix} 1 & 0 \\ \lambda_j & 1 \end{pmatrix}, & \text{if } 2 \nmid j, \\ \begin{pmatrix} 1 & 0 \\ \lambda_j & 0 \end{pmatrix}, & \text{otherwise.} \end{cases}
 \end{aligned}$$

Thus the repair bandwidth γ_i of node i is

$$\gamma_i = \sum_{j=0, j \neq i}^{n-1} \text{rank}(S + S'A_j) = \left(\frac{n}{2} - 1\right) \times 2 + \frac{n}{2} = \frac{3n}{2} - 2,$$

while the rebuilding access Γ_i of node i is

$$\Gamma_i = \sum_{j=0, j \neq i}^{n-1} N_c(S + S'A_j) = \left(\frac{n}{2} - 1\right) \times 2 + \frac{n}{2} = \frac{3n}{2} - 2.$$

This completes the proof. \square

Remark 2. When n is odd in Construction 3, similar to the proof of Theorem 2, we have that both the repair bandwidth and the rebuilding access of node i ($i \in [0, n)$) in local group j ($j \in [0, \mu)$) of the array code in Construction 3 are

$$\gamma_i = \Gamma_i = \begin{cases} \frac{3n-3}{2}, & \text{if } 2 \mid i, \\ \frac{3n-5}{2}, & \text{otherwise.} \end{cases}$$

Theorem 3. Let q be a prime power such that there exists a multiplicative subgroup G of $\mathbb{F}_q \setminus \{0\}$ of size at least n and with at least μ cosets. Choosing $\lambda_{i,t}$, $i \in [0, n)$, $t \in [0, r)$ from G , then the code in Construction 3 is a PMDS array code if $\theta_0, \dots, \theta_{\mu-1}$ are elements from distinct cosets of G .

Proof. By (14) and the fact that $\lambda_0, \dots, \lambda_{n-1}$ are pairwise distinct nonzero elements in \mathbb{F}_q , we easily have that $A_i - A_j$ and therefore $\begin{pmatrix} I & I \\ A_i & A_j \end{pmatrix}$ are nonsingular for any $i, j \in [0, n)$ with $i \neq j$, therefore, i) of Definition 1 is satisfied.

Now let us check ii) of Definition 1. Suppose there are two failed nodes in every local group and two more anywhere. Similar to the proof of Theorem 1, we only need to analyze the following two cases:

- Both the two extra failed nodes are in the same local group, say group i . Assume that nodes $in + j_0, \dots, in + j_3$ are failed, where $0 \leq j_0 < \dots < j_3 < n$. Let $J = \{j_0, \dots, j_3\}$, then the original file can be reconstructed if the matrix $\begin{pmatrix} H_i[J] \\ P_i[J] \end{pmatrix}$ is of full rank.

Note that

$$\begin{aligned}
 & \begin{pmatrix} H_i[J] \\ P_i[J] \end{pmatrix} \\
 &= \begin{pmatrix} I & I & I & I \\ A_{j_0} & A_{j_1} & A_{j_2} & A_{j_3} \\ \lambda_{j_0}^2 I & \lambda_{j_1}^2 I & \lambda_{j_2}^2 I & \lambda_{j_3}^2 I \\ \theta_i \lambda_{j_0}^{-1} I & \theta_i \lambda_{j_1}^{-1} I & \theta_i \lambda_{j_2}^{-1} I & \theta_i \lambda_{j_3}^{-1} I \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \lambda_{j_0} & x_0 & \lambda_{j_1} & x_1 & \lambda_{j_2} & x_2 & \lambda_{j_3} & x_3 \\ 0 & \lambda_{j_0} & 0 & \lambda_{j_1} & 0 & \lambda_{j_2} & 0 & \lambda_{j_3} \\ \lambda_{j_0}^2 & 0 & \lambda_{j_1}^2 & 0 & \lambda_{j_2}^2 & 0 & \lambda_{j_3}^2 & 0 \\ 0 & \lambda_{j_0}^2 & 0 & \lambda_{j_1}^2 & 0 & \lambda_{j_2}^2 & 0 & \lambda_{j_3}^2 \\ \theta_i \lambda_{j_0}^{-1} & 0 & \theta_i \lambda_{j_1}^{-1} & 0 & \theta_i \lambda_{j_2}^{-1} & 0 & \theta_i \lambda_{j_3}^{-1} & 0 \\ 0 & \theta_i \lambda_{j_0}^{-1} & 0 & \theta_i \lambda_{j_1}^{-1} & 0 & \theta_i \lambda_{j_2}^{-1} & 0 & \theta_i \lambda_{j_3}^{-1} \end{pmatrix},
 \end{aligned}$$

where $x_j = 0, 1$. By swapping the rows and columns, it is easy to see that $\begin{pmatrix} H_i[J] \\ P_i[J] \end{pmatrix}$ is equivalent to the following matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \lambda_{j_0} & \lambda_{j_1} & \lambda_{j_2} & \lambda_{j_3} & x_0 & x_1 & x_2 & x_3 \\ \lambda_{j_0}^2 & \lambda_{j_1}^2 & \lambda_{j_2}^2 & \lambda_{j_3}^2 & 0 & 0 & 0 & 0 \\ \theta_i \lambda_{j_0}^{-1} & \theta_i \lambda_{j_1}^{-1} & \theta_i \lambda_{j_2}^{-1} & \theta_i \lambda_{j_3}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \lambda_{j_0} & \lambda_{j_1} & \lambda_{j_2} & \lambda_{j_3} \\ 0 & 0 & 0 & 0 & \lambda_{j_0}^2 & \lambda_{j_1}^2 & \lambda_{j_2}^2 & \lambda_{j_3}^2 \\ 0 & 0 & 0 & 0 & \theta_i \lambda_{j_0}^{-1} & \theta_i \lambda_{j_1}^{-1} & \theta_i \lambda_{j_2}^{-1} & \theta_i \lambda_{j_3}^{-1} \end{pmatrix}, \quad (20)$$

under elementary transformation. The matrix in (20) is nonsingular if

$$\Lambda = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda_{j_0} & \lambda_{j_1} & \lambda_{j_2} & \lambda_{j_3} \\ \lambda_{j_0}^2 & \lambda_{j_1}^2 & \lambda_{j_2}^2 & \lambda_{j_3}^2 \\ \theta_i \lambda_{j_0}^{-1} & \theta_i \lambda_{j_1}^{-1} & \theta_i \lambda_{j_2}^{-1} & \theta_i \lambda_{j_3}^{-1} \end{pmatrix} \quad (21)$$

is nonsingular. By $\theta_i \neq 0$, we have

$$\begin{aligned}
 & \begin{pmatrix} & & \theta_i^{-1} & \\ 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \Lambda \begin{pmatrix} \lambda_{j_0} & & & \\ & \lambda_{j_1} & & \\ & & \lambda_{j_2} & \\ & & & \lambda_{j_3} \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda_{j_0} & \lambda_{j_1} & \lambda_{j_2} & \lambda_{j_3} \\ \lambda_{j_0}^2 & \lambda_{j_1}^2 & \lambda_{j_2}^2 & \lambda_{j_3}^2 \\ \lambda_{j_0}^3 & \lambda_{j_1}^3 & \lambda_{j_2}^3 & \lambda_{j_3}^3 \end{pmatrix},
 \end{aligned}$$

which is the transpose of a Vandermonde matrix and has the same rank as the matrix Λ in (21), in conjunction with the facts that $\lambda_{j_0}, \lambda_{j_1}, \lambda_{j_2}, \lambda_{j_3}$ are pairwise distinct nonzero elements, we conclude that the matrix $\begin{pmatrix} H_i[J] \\ P_i[J] \end{pmatrix}$ is nonsingular.

- Suppose that the two extra failed nodes are in two different local groups, say group i and group k , where $0 \leq i < k < \mu$. Assume that nodes $in + j_0, in + j_1, in + j_2$ and nodes $kn + t_0, kn + t_1, kn + t_2$ are failed, where $0 \leq j_0 < j_1 < j_2 < n$ and $0 \leq t_0 < t_1 < t_2 < n$. Let

$$\hat{\mathcal{H}} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_{j_0} & x_0 & \lambda_{j_1} & x_1 & \lambda_{j_2} & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_{j_0} & 0 & \lambda_{j_1} & 0 & \lambda_{j_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{t_0} & x_3 & \lambda_{t_1} & x_4 & \lambda_{t_2} & x_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{t_0} & 0 & \lambda_{t_1} & 0 & \lambda_{t_2} & 0 \\ \lambda_{j_0}^2 & 0 & \lambda_{j_1}^2 & 0 & \lambda_{j_2}^2 & 0 & \lambda_{t_0}^2 & 0 & \lambda_{t_1}^2 & 0 & \lambda_{t_2}^2 & 0 & 0 \\ 0 & \lambda_{j_0}^2 & 0 & \lambda_{j_1}^2 & 0 & \lambda_{j_2}^2 & 0 & \lambda_{t_0}^2 & 0 & \lambda_{t_1}^2 & 0 & \lambda_{t_2}^2 & 0 \\ \theta_i \lambda_{j_0}^{-1} & 0 & \theta_i \lambda_{j_1}^{-1} & 0 & \theta_i \lambda_{j_2}^{-1} & 0 & \theta_k \lambda_{t_0}^{-1} & 0 & \theta_k \lambda_{t_1}^{-1} & 0 & \theta_k \lambda_{t_2}^{-1} & 0 & 0 \\ 0 & \theta_i \lambda_{j_0}^{-1} & 0 & \theta_i \lambda_{j_1}^{-1} & 0 & \theta_i \lambda_{j_2}^{-1} & 0 & \theta_k \lambda_{j_0}^{-1} & 0 & \theta_k \lambda_{j_1}^{-1} & 0 & \theta_k \lambda_{j_2}^{-1} & 0 \end{pmatrix}. \quad (17)$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_{j_0} & \lambda_{j_1} & \lambda_{j_2} & 0 & 0 & 0 & x_0 & x_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{t_0} & \lambda_{t_1} & \lambda_{t_2} & 0 & 0 & 0 & x_3 & x_4 & x_5 & 0 \\ \lambda_{j_0}^2 & \lambda_{j_1}^2 & \lambda_{j_2}^2 & \lambda_{t_0}^2 & \lambda_{t_1}^2 & \lambda_{t_2}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_i \lambda_{j_0}^{-1} & \theta_i \lambda_{j_1}^{-1} & \theta_i \lambda_{j_2}^{-1} & \theta_k \lambda_{t_0}^{-1} & \theta_k \lambda_{t_1}^{-1} & \theta_k \lambda_{t_2}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{j_0} & \lambda_{j_1} & \lambda_{j_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{t_0} & \lambda_{t_1} & \lambda_{t_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{j_0}^2 & \lambda_{j_1}^2 & \lambda_{j_2}^2 & \lambda_{t_0}^2 & \lambda_{t_1}^2 & \lambda_{t_2}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \theta_i \lambda_{j_0}^{-1} & \theta_i \lambda_{j_1}^{-1} & \theta_i \lambda_{j_2}^{-1} & \theta_k \lambda_{t_0}^{-1} & \theta_k \lambda_{t_1}^{-1} & \theta_k \lambda_{t_2}^{-1} & 0 \end{pmatrix}. \quad (18)$$

$$\det(B) = \det \begin{pmatrix} \det \begin{pmatrix} 1 & 1 & 1 \\ \lambda_{j_0} & \lambda_{j_1} & \lambda_{j_2} \\ \lambda_{j_0}^2 & \lambda_{j_1}^2 & \lambda_{j_2}^2 \end{pmatrix} & \det \begin{pmatrix} 1 & 1 & 1 \\ \lambda_{t_0} & \lambda_{t_1} & \lambda_{t_2} \\ \lambda_{t_0}^2 & \lambda_{t_1}^2 & \lambda_{t_2}^2 \end{pmatrix} \\ \det \begin{pmatrix} 1 & 1 & 1 \\ \lambda_{j_0} & \lambda_{j_1} & \lambda_{j_2} \\ \theta_i \lambda_{j_0}^{-1} & \theta_i \lambda_{j_1}^{-1} & \theta_i \lambda_{j_2}^{-1} \end{pmatrix} & \det \begin{pmatrix} 1 & 1 & 1 \\ \lambda_{t_0} & \lambda_{t_1} & \lambda_{t_2} \\ \theta_k \lambda_{t_0}^{-1} & \theta_k \lambda_{t_1}^{-1} & \theta_k \lambda_{t_2}^{-1} \end{pmatrix} \end{pmatrix}. \quad (19)$$

$J = \{j_0, j_1, j_2\}$ and $T = \{t_0, t_1, t_2\}$, then the original file can be reconstructed if the following matrix

$$\hat{\mathcal{H}} = \begin{pmatrix} H_i[J] & H_k[T] \\ P_i[J] & P_k[T] \\ A_{j_0} & A_{j_1} & A_{j_2} \\ \lambda_{j_0}^2 I & \lambda_{j_1}^2 I & \lambda_{j_2}^2 I \\ \theta_i \lambda_{j_0}^{-1} I & \theta_i \lambda_{j_1}^{-1} I & \theta_i \lambda_{j_2}^{-1} I \end{pmatrix} \begin{pmatrix} A_{t_0} & A_{t_1} & A_{t_2} \\ \lambda_{t_0}^2 I & \lambda_{t_1}^2 I & \lambda_{t_2}^2 I \\ \theta_k \lambda_{t_0}^{-1} I & \theta_k \lambda_{t_1}^{-1} I & \theta_k \lambda_{t_2}^{-1} I \end{pmatrix}$$

is nonsingular. Substituting (14) into $\hat{\mathcal{H}}$, we have (17), where $x_0, \dots, x_5 = 0$ or 1. By swapping the rows and columns of $\hat{\mathcal{H}}$ in (17), it is easy to see that $\hat{\mathcal{H}}$ is equivalent to the matrix in (18) under elementary transformation,

which is nonsingular if the following matrix

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ \lambda_{j_0} & \lambda_{j_1} & \lambda_{j_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \lambda_{t_0} & \lambda_{t_1} & \lambda_{t_2} \\ \lambda_{j_0}^2 & \lambda_{j_1}^2 & \lambda_{j_2}^2 & \lambda_{t_0}^2 & \lambda_{t_1}^2 & \lambda_{t_2}^2 \\ \theta_i \lambda_{j_0}^{-1} & \theta_i \lambda_{j_1}^{-1} & \theta_i \lambda_{j_2}^{-1} & \theta_k \lambda_{t_0}^{-1} & \theta_k \lambda_{t_1}^{-1} & \theta_k \lambda_{t_2}^{-1} \end{pmatrix}$$

is nonsingular.

By Lemma 3, we have (19). By factoring out the nonzero Vandermonde determinant from each column of the determinant in (19), we further have

$$\det(B) \neq 0 \iff \det \begin{pmatrix} \lambda_{j_0} \lambda_{j_1} \lambda_{j_2} & \lambda_{t_0} \lambda_{t_1} \lambda_{t_2} \\ \theta_i & \theta_k \end{pmatrix} \neq 0,$$

where the last inequality holds since both $\lambda_{j_0} \lambda_{j_1} \lambda_{j_2}$ and $\lambda_{t_0} \lambda_{t_1} \lambda_{t_2}$ are in the subgroup G , while θ_i and θ_k are in different cosets of G .

Therefore, B and thus $\hat{\mathcal{H}}$ is nonsingular, and the original file can be reconstructed.

This finishes the proof. \square

Remark 3. Although Construction 3 only supports two local parities, it has the following novelty.

- It is the first time to use non-diagonal matrices (cf. (14)) as building blocks in PMDS array codes besides diagonal matrices, i.e., it is not required that each sub-stripe of the local code be a scalar MDS code as in [38], which is the key to reduce the rebuilding access.
- For the local code of the new PMDS array code in Construction 3, its rebuilding access is around 0.75 (more precisely, $\frac{3n/2-2}{2n-4}$) times that of a Reed-Solomon code with the same parameters. In [49], a tight lower bound of the average rebuilding access of $[n, n-2]$ MDS array codes with sub-packetization level 2 as well as the optimal code construction was derived. Although the exact expression of the general lower bound is complicated, it was shown that the average rebuilding access of $[n, n-2]$ MDS array codes with sub-packetization level 2 is larger than 0.72 times that of an RS code with the same parameters for $n \leq 50$. This shows that the local code of the new PMDS array code in Construction 3 that we choose has rebuilding access which is about 1.04 times the lower bound in [49] for $n \leq 50$. Of course, we can choose an MDS array code with the optimal average rebuilding access in [49] as the local code. However, it will be very difficult to verify ii) of Definition 1, which will be left as our future research.

IV. A NEW $(\mu, n; r, s = 3)$ PMDS ARRAY CODE CONSTRUCTION

In this subsection, we propose a new $(\mu, n; r, s = 3)$ PMDS array code construction, where each P_i in (1) has three block rows. To prove that an array code defined by the parity-check matrix in (1) is a PMDS array code, similar to the previous sections, we need to calculate the determinants of the sub-block matrices that are formed by any $r+3$ thick columns of $\begin{pmatrix} H_i \\ P_i \end{pmatrix}$, any $r+2$ thick columns of $\begin{pmatrix} H_i \\ P_i^{[T]} \end{pmatrix}$ ($T \subset [0, 3)$, $|T| = 2$), and any $r+1$ thick columns of $\begin{pmatrix} H_i \\ P_i^{[t]} \end{pmatrix}$ ($t = 0, 1, 2$). If we define P_i similar to that in (9), e.g., by adding a block row

$$\begin{pmatrix} A_0^{r+1} & A_1^{r+1} & \cdots & A_{n-1}^{r+1} \end{pmatrix}$$

or

$$\begin{pmatrix} \theta'_i A_0^{-2} & \theta'_i A_1^{-2} & \cdots & \theta'_i A_{n-1}^{-2} \end{pmatrix},$$

then some of the sub-block matrices mentioned above will not be equivalent to a block Vandermonde matrix anymore, and their determinants will be hard to calculate. By defining P_i as ℓ matrices of order $3 \times \ell$, with each $3 \times \ell$ matrix being a sub-matrix of a Cauchy-Vandermonde matrix, the above concern can be addressed by applying Lemma 5.

Following the notation in Construction 1, we define two variants of the matrices A_i ($i \in [0, n)$) in the following. For $i \in [0, n)$, let A'_i and A''_i be $\ell \times \ell$ matrices that satisfy

$$\begin{pmatrix} V_{\bar{i},0} \\ V_{\bar{i},1} \\ \vdots \\ V_{\bar{i},r-1} \end{pmatrix} A'_i = \begin{pmatrix} \frac{1}{\lambda_{i,0}-d_0} V_{\bar{i},0} \\ \frac{1}{\lambda_{i,1}-d_0} V_{\bar{i},1} \\ \vdots \\ \frac{1}{\lambda_{i,r-1}-d_0} V_{\bar{i},r-1} \end{pmatrix}, \quad (22)$$

and

$$\begin{pmatrix} V_{\bar{i},0} \\ V_{\bar{i},1} \\ \vdots \\ V_{\bar{i},r-1} \end{pmatrix} A''_i = \begin{pmatrix} \frac{1}{\lambda_{i,0}-d_1} V_{\bar{i},0} \\ \frac{1}{\lambda_{i,1}-d_1} V_{\bar{i},1} \\ \vdots \\ \frac{1}{\lambda_{i,r-1}-d_1} V_{\bar{i},r-1} \end{pmatrix}, \quad (23)$$

where $d_0, d_1 \in \mathbf{F}_q \setminus \{\lambda_{i,t}, i \in [0, n), t \in [0, r)\}$ and $d_0 \neq d_1$.

Similarly, we have that A'_i and A''_i are diagonal matrices and the a -th rows of A'_i and A''_i are

$$e_a A'_i = \frac{1}{\lambda_{i,a_{\bar{i}}}-d_0} e_a \quad \text{and} \quad e_a A''_i = \frac{1}{\lambda_{i,a_{\bar{i}}}-d_1} e_a \quad (24)$$

respectively, where $i \in [0, n)$, $a \in [0, \ell)$, and $a_{\bar{i}}$ denotes the \bar{i} -th element in the r -ary expansion of a .

Construction 4. Let μ, r, n', n be four positive integers, where $\mu, r \geq 2$ and $n > n'$, and let $\ell = r^{n'}$. We construct a new $[\mu n, \mu(n-r) - 3, \ell]$ array code over \mathbf{F}_q with the parity-check matrix having the form as in (1) with

$$H_i = H, \quad (25)$$

and

$$P_i = \begin{pmatrix} A_0^r & A_1^r & \cdots & A_{n-1}^r \\ \theta_i A'_0 & \theta_i A'_1 & \cdots & \theta_i A'_{n-1} \\ \delta_i A''_0 & \delta_i A''_1 & \cdots & \delta_i A''_{n-1} \end{pmatrix} \quad (26)$$

for $i \in [0, \mu)$, where H is defined in (3), i.e., the parity-check matrix of the code in Construction 1, $A_i, A'_i,$ and A''_i are $\ell \times \ell$ matrices defined in (4), (22), and (23), respectively, $\theta_i, \delta_i \in \mathbf{F}_q \setminus \{0\}$ for $i \in [0, \mu)$.

Theorem 4. The code in Construction 4 is a $(\mu, n; r, s = 3)$ PMDS array code over \mathbf{F}_q with sub-packetization level ℓ if the following conditions C1–C4 hold.

- C1. $q = q_0^3$, where $q_0 > \mu(\Phi + 1)$ is a prime power such that there exists a multiplicative subgroup G of $\mathbf{F}_{q_0} \setminus \{0\}$ of size at least $\Phi + 1$ and with at least μ cosets, where Φ is defined in (7).
- C2. d_0 and d_1 are two distinct elements chosen from \mathbf{F}_{q_0} . $\lambda_{i,t}, i \in [0, n), t \in [0, r)$ are chosen from $\Omega \setminus \{d_0\}$ such that R1-R3 of Lemma 6 hold, where

$$\Omega = \{\lambda \in \mathbf{F}_{q_0} : \frac{1}{\lambda - d_1} \in G\},$$

and clearly $|\Omega| \geq \Phi + 1$.

- C3. $\Theta = \{\theta_0, \theta_1, \dots, \theta_{\mu-1}\} \subset \mathbf{F}_q \setminus \{0\}$ is 3-wise independent over \mathbf{F}_{q_0} , i.e., any t -subset of Θ with $t \leq 3$ is linearly independent over \mathbf{F}_{q_0} .
- C4. $\delta_0, \delta_1, \dots, \delta_{\mu-1} \in \mathbf{F}_{q_0}$ are elements from distinct cosets of G .

$$B_a = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{j_0, a_{j_0}^-} & \lambda_{j_1, a_{j_1}^-} & \cdots & \lambda_{j_r, a_{j_r}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j_0, a_{j_0}^{r-1}} & \lambda_{j_1, a_{j_1}^{r-1}} & \cdots & \lambda_{j_r, a_{j_r}^{r-1}} \\ & 1 & 1 & \cdots & 1 \\ & \lambda_{t_0, a_{t_0}^-} & \lambda_{t_1, a_{t_1}^-} & \cdots & \lambda_{t_{r+1}, a_{t_{r+1}}^-} \\ & \vdots & \vdots & \ddots & \vdots \\ & \lambda_{t_0, a_{t_0}^{r-1}} & \lambda_{t_1, a_{t_1}^{r-1}} & \cdots & \lambda_{t_{r+1}, a_{t_{r+1}}^{r-1}} \\ \frac{\lambda_{j_0, a_{j_0}^-}^r}{\theta_i} & \frac{\lambda_{j_1, a_{j_1}^-}^r}{\theta_i} & \cdots & \frac{\lambda_{j_r, a_{j_r}^-}^r}{\theta_i} & \frac{\lambda_{t_0, a_{t_0}^-}^r}{\theta_k} & \frac{\lambda_{t_1, a_{t_1}^-}^r}{\theta_k} & \cdots & \frac{\lambda_{t_{r+1}, a_{t_{r+1}}^-}^r}{\theta_k} \\ \frac{\lambda_{j_0, a_{j_0}^-}^r}{\delta_i} & \frac{\lambda_{j_1, a_{j_1}^-}^r}{\delta_i} & \cdots & \frac{\lambda_{j_r, a_{j_r}^-}^r}{\delta_i} & \frac{\lambda_{t_0, a_{t_0}^-}^r}{\delta_k} & \frac{\lambda_{t_1, a_{t_1}^-}^r}{\delta_k} & \cdots & \frac{\lambda_{t_{r+1}, a_{t_{r+1}}^-}^r}{\delta_k} \\ \frac{\lambda_{j_0, a_{j_0}^-}^r}{\lambda_{j_0, a_{j_0}^-} - d_0} & \frac{\lambda_{j_1, a_{j_1}^-}^r}{\lambda_{j_1, a_{j_1}^-} - d_0} & \cdots & \frac{\lambda_{j_r, a_{j_r}^-}^r}{\lambda_{j_r, a_{j_r}^-} - d_0} & \frac{\lambda_{t_0, a_{t_0}^-}^r}{\lambda_{t_0, a_{t_0}^-} - d_0} & \frac{\lambda_{t_1, a_{t_1}^-}^r}{\lambda_{t_1, a_{t_1}^-} - d_0} & \cdots & \frac{\lambda_{t_{r+1}, a_{t_{r+1}}^-}^r}{\lambda_{t_{r+1}, a_{t_{r+1}}^-} - d_0} \\ \frac{\lambda_{j_0, a_{j_0}^-}^r}{\lambda_{j_0, a_{j_0}^-} - d_1} & \frac{\lambda_{j_1, a_{j_1}^-}^r}{\lambda_{j_1, a_{j_1}^-} - d_1} & \cdots & \frac{\lambda_{j_r, a_{j_r}^-}^r}{\lambda_{j_r, a_{j_r}^-} - d_1} & \frac{\lambda_{t_0, a_{t_0}^-}^r}{\lambda_{t_0, a_{t_0}^-} - d_1} & \frac{\lambda_{t_1, a_{t_1}^-}^r}{\lambda_{t_1, a_{t_1}^-} - d_1} & \cdots & \frac{\lambda_{t_{r+1}, a_{t_{r+1}}^-}^r}{\lambda_{t_{r+1}, a_{t_{r+1}}^-} - d_1} \end{pmatrix}, \quad a \in [0, \ell). \quad (27)$$

$$K_a = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{j_0, a_{j_0}^-} & \lambda_{j_1, a_{j_1}^-} & \cdots & \lambda_{j_r, a_{j_r}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j_0, a_{j_0}^r} & \lambda_{j_1, a_{j_1}^r} & \cdots & \lambda_{j_r, a_{j_r}^r} \end{pmatrix} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{t_0, a_{t_0}^-} & \lambda_{t_1, a_{t_1}^-} & \cdots & \lambda_{t_{r+1}, a_{t_{r+1}}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{t_0, a_{t_0}^{r-1}} & \lambda_{t_1, a_{t_1}^{r-1}} & \cdots & \lambda_{t_{r+1}, a_{t_{r+1}}^{r-1}} \\ \frac{\theta_k}{\lambda_{t_0, a_{t_0}^-} - d_0} & \frac{\theta_k}{\lambda_{t_1, a_{t_1}^-} - d_0} & \cdots & \frac{\theta_k}{\lambda_{t_{r+1}, a_{t_{r+1}}^-} - d_0} \\ \frac{\delta_k}{\lambda_{t_0, a_{t_0}^-} - d_1} & \frac{\delta_k}{\lambda_{t_1, a_{t_1}^-} - d_1} & \cdots & \frac{\delta_k}{\lambda_{t_{r+1}, a_{t_{r+1}}^-} - d_1} \end{pmatrix} \\ - \theta_i \delta_k \det \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{j_0, a_{j_0}^-} & \lambda_{j_1, a_{j_1}^-} & \cdots & \lambda_{j_r, a_{j_r}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j_0, a_{j_0}^{r-1}} & \lambda_{j_1, a_{j_1}^{r-1}} & \cdots & \lambda_{j_r, a_{j_r}^{r-1}} \\ \frac{1}{\lambda_{j_0, a_{j_0}^-} - d_0} & \frac{1}{\lambda_{j_1, a_{j_1}^-} - d_0} & \cdots & \frac{1}{\lambda_{j_r, a_{j_r}^-} - d_0} \end{pmatrix}}_A \det \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{t_0, a_{t_0}^-} & \lambda_{t_1, a_{t_1}^-} & \cdots & \lambda_{t_r, a_{t_r}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{t_0, a_{t_0}^r} & \lambda_{t_1, a_{t_1}^r} & \cdots & \lambda_{t_{r+1}, a_{t_{r+1}}^r} \\ \frac{1}{\lambda_{t_0, a_{t_0}^-} - d_1} & \frac{1}{\lambda_{t_1, a_{t_1}^-} - d_1} & \cdots & \frac{1}{\lambda_{t_{r+1}, a_{t_{r+1}}^-} - d_1} \end{pmatrix}}_B \\ + \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{j_0, a_{j_0}^-} & \lambda_{j_1, a_{j_1}^-} & \cdots & \lambda_{j_r, a_{j_r}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j_0, a_{j_0}^r} & \lambda_{j_1, a_{j_1}^r} & \cdots & \lambda_{j_r, a_{j_r}^r} \\ \frac{\delta_i}{\lambda_{j_0, a_{j_0}^-} - d_1} & \frac{\delta_i}{\lambda_{j_1, a_{j_1}^-} - d_1} & \cdots & \frac{\delta_i}{\lambda_{j_r, a_{j_r}^-} - d_1} \end{pmatrix} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{t_0, a_{t_0}^-} & \lambda_{t_1, a_{t_1}^-} & \cdots & \lambda_{t_r, a_{t_r}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{t_0, a_{t_0}^r} & \lambda_{t_1, a_{t_1}^r} & \cdots & \lambda_{t_{r+1}, a_{t_{r+1}}^r} \\ \frac{\theta_k}{\lambda_{t_0, a_{t_0}^-} - d_0} & \frac{\theta_k}{\lambda_{t_1, a_{t_1}^-} - d_0} & \cdots & \frac{\theta_k}{\lambda_{t_{r+1}, a_{t_{r+1}}^-} - d_0} \end{pmatrix} \\ \neq 0. \quad (28)$$

transposing). By C3, this linear combination cannot be zero. Therefore, $\text{blkdiag}(B_0, B_1, \dots, B_{\ell-1})$ and thus $\hat{\mathcal{H}}$ is nonsingular, and the original file can be reconstructed.

- The three extra node failures are in three distinct groups, say in groups i , k , and l , and assume that nodes $in + j_0, \dots, in + j_r$, nodes $kn + t_0, \dots, kn + t_r$, and nodes $ln + u_0, \dots, ln + u_r$ are failed, where $0 \leq i < k < l < \mu$, $0 \leq j_0 < \dots < j_r < n$, $0 \leq t_0 < \dots < t_r < n$, and $0 \leq u_0 < \dots < u_r < n$. Let $J = \{j_0, \dots, j_r\}$, $T = \{t_0, \dots, t_r\}$, and $U = \{u_0, \dots, u_r\}$, then the original file can be reconstructed if the matrix $\hat{\mathcal{H}}$ in (29) is nonsingular.

Similarly, by swapping the rows and columns of $\hat{\mathcal{H}}$, $\hat{\mathcal{H}}$ is

equivalent to

$$\text{blkdiag}(B_0, B_1, \dots, B_{\ell-1})$$

under elementary transformation, where B_a is formed by the $a, a + \ell, \dots, (a + (3r + 2)\ell)$ -th rows and the $a, a + \ell, \dots, (a + (3r + 2)\ell)$ -th columns of $\hat{\mathcal{H}}$ for $a \in [0, \ell)$, and can be expressed as in (30) according to (5).

By Lemma 3, $\det(B_a) \neq 0$ is equivalent to the inequality in (31) for all $a \in [0, \ell)$, where we also use $|A|$ to denote the determinant of the matrix A . By Lemma 5, we further have

$$K_a = D_0 D_1 D_2 K'_a,$$

$$\hat{\mathcal{H}} = \begin{pmatrix} H_i[J] & & & \\ & H_k[T] & & \\ & & H_l[U] & \\ P_i[J] & P_k[T] & P_l[U] & \end{pmatrix} = \begin{pmatrix} I & I & \cdots & I \\ A_{j_0} & A_{j_1} & \cdots & A_{j_r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{j_0}^{r-1} & A_{j_1}^{r-1} & \cdots & A_{j_r}^{r-1} \\ & I & I & \cdots & I \\ & A_{t_0} & A_{t_1} & \cdots & A_{t_r} \\ & \vdots & \vdots & \ddots & \vdots \\ & A_{t_0}^{r-1} & A_{t_1}^{r-1} & \cdots & A_{t_r}^{r-1} \\ & & I & I & \cdots & I \\ & & A_{u_0} & A_{u_1} & \cdots & A_{u_r} \\ & & \vdots & \vdots & \ddots & \vdots \\ & & A_{u_0}^{r-1} & A_{u_1}^{r-1} & \cdots & A_{u_r}^{r-1} \\ A_{j_0}^r & A_{j_1}^r & \cdots & A_{j_r}^r & A_{t_0}^r & A_{t_1}^r & \cdots & A_{t_r}^r & A_{u_0}^r & A_{u_1}^r & \cdots & A_{u_r}^r \\ \theta_i A'_{j_0} & \theta_i A'_{j_1} & \cdots & \theta_i A'_{j_r} & \theta_k A'_{t_0} & \theta_k A'_{t_1} & \cdots & \theta_k A'_{t_r} & \theta_l A'_{u_0} & \theta_l A'_{u_1} & \cdots & \theta_l A'_{u_r} \\ \delta_i A''_{j_0} & \delta_i A''_{j_1} & \cdots & \delta_i A''_{j_r} & \delta_k A''_{t_0} & \delta_k A''_{t_1} & \cdots & \delta_k A''_{t_r} & \delta_k A''_{u_0} & \delta_k A''_{u_1} & \cdots & \delta_k A''_{u_r} \end{pmatrix}. \quad (29)$$

$$B_a = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{j_0, a_{j_0}} & \lambda_{j_1, a_{j_1}} & \cdots & \lambda_{j_r, a_{j_r}} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j_0, a_{j_0}}^{r-1} & \lambda_{j_1, a_{j_1}}^{r-1} & \cdots & \lambda_{j_r, a_{j_r}}^{r-1} \\ & 1 & 1 & \cdots & 1 \\ & \lambda_{t_0, a_{t_0}} & \lambda_{t_1, a_{t_1}} & \cdots & \lambda_{t_r, a_{t_r}} \\ & \vdots & \vdots & \ddots & \vdots \\ & \lambda_{t_0, a_{t_0}}^{r-1} & \lambda_{t_1, a_{t_1}}^{r-1} & \cdots & \lambda_{t_r, a_{t_r}}^{r-1} \\ & & 1 & 1 & \cdots & 1 \\ & & \lambda_{u_0, a_{u_0}} & \lambda_{u_1, a_{u_1}} & \cdots & \lambda_{u_r, a_{u_r}} \\ & & \vdots & \vdots & \ddots & \vdots \\ & & \lambda_{u_0, a_{u_0}}^{r-1} & \lambda_{u_1, a_{u_1}}^{r-1} & \cdots & \lambda_{u_r, a_{u_r}}^{r-1} \\ \frac{\lambda_{j_0, a_{j_0}}^r}{\lambda_{j_0, a_{j_0}} - d_0} & \frac{\lambda_{j_1, a_{j_1}}^r}{\lambda_{j_1, a_{j_1}} - d_0} & \cdots & \frac{\lambda_{j_r, a_{j_r}}^r}{\lambda_{j_r, a_{j_r}} - d_0} & \frac{\lambda_{t_0, a_{t_0}}^r}{\lambda_{t_0, a_{t_0}} - d_0} & \frac{\lambda_{t_1, a_{t_1}}^r}{\lambda_{t_1, a_{t_1}} - d_0} & \cdots & \frac{\lambda_{t_r, a_{t_r}}^r}{\lambda_{t_r, a_{t_r}} - d_0} & \frac{\lambda_{u_0, a_{u_0}}^r}{\lambda_{u_0, a_{u_0}} - d_0} & \frac{\lambda_{u_1, a_{u_1}}^r}{\lambda_{u_1, a_{u_1}} - d_0} & \cdots & \frac{\lambda_{u_r, a_{u_r}}^r}{\lambda_{u_r, a_{u_r}} - d_0} \\ \frac{\theta_i}{\lambda_{j_0, a_{j_0}} - d_1} & \frac{\theta_i}{\lambda_{j_1, a_{j_1}} - d_1} & \cdots & \frac{\theta_i}{\lambda_{j_r, a_{j_r}} - d_1} & \frac{\theta_k}{\lambda_{t_0, a_{t_0}} - d_1} & \frac{\theta_k}{\lambda_{t_1, a_{t_1}} - d_1} & \cdots & \frac{\theta_k}{\lambda_{t_r, a_{t_r}} - d_1} & \frac{\theta_l}{\lambda_{u_0, a_{u_0}} - d_1} & \frac{\theta_l}{\lambda_{u_1, a_{u_1}} - d_1} & \cdots & \frac{\theta_l}{\lambda_{u_r, a_{u_r}} - d_1} \\ \frac{\delta_i}{\lambda_{j_0, a_{j_0}} - d_1} & \frac{\delta_i}{\lambda_{j_1, a_{j_1}} - d_1} & \cdots & \frac{\delta_i}{\lambda_{j_r, a_{j_r}} - d_1} & \frac{\delta_k}{\lambda_{t_0, a_{t_0}} - d_1} & \frac{\delta_k}{\lambda_{t_1, a_{t_1}} - d_1} & \cdots & \frac{\delta_k}{\lambda_{t_r, a_{t_r}} - d_1} & \frac{\delta_l}{\lambda_{u_0, a_{u_0}} - d_1} & \frac{\delta_l}{\lambda_{u_1, a_{u_1}} - d_1} & \cdots & \frac{\delta_l}{\lambda_{u_r, a_{u_r}} - d_1} \end{pmatrix}. \quad (30)$$

where

$$D_0 = \prod_{0 \leq l < i \leq r} (\lambda_{j_i, a_{j_i}} - \lambda_{j_l, a_{j_l}}),$$

$$D_1 = \prod_{0 \leq l < i \leq r} (\lambda_{t_i, a_{t_i}} - \lambda_{t_l, a_{t_l}}),$$

$$D_2 = \prod_{0 \leq l < i \leq r} (\lambda_{u_i, a_{u_i}} - \lambda_{u_l, a_{u_l}}),$$

K'_a

$$= \begin{pmatrix} \frac{1}{\prod_{i=0}^r (\lambda_{j_i, a_{j_i}} - d_0)} & \frac{1}{\prod_{i=0}^r (\lambda_{t_i, a_{t_i}} - d_0)} & \frac{1}{\prod_{i=0}^r (\lambda_{u_i, a_{u_i}} - d_0)} \\ \frac{\delta_i}{\prod_{i=0}^r (\lambda_{j_i, a_{j_i}} - d_1)} & \frac{\delta_k}{\prod_{i=0}^r (\lambda_{t_i, a_{t_i}} - d_1)} & \frac{\delta_l}{\prod_{i=0}^r (\lambda_{u_i, a_{u_i}} - d_1)} \end{pmatrix}.$$

By writing the Laplace expansion of the above determinant K'_a over the second row, K'_a can be expressed as a linear combination of θ_i , θ_k , and θ_l . The coefficient of θ_i is

$$\frac{1}{\prod_{i=0}^r (\lambda_{j_i, a_{j_i}} - d_0)} \times \left(\frac{\delta_k}{\prod_{i=0}^r (\lambda_{t_i, a_{t_i}} - d_1)} - \frac{\delta_l}{\prod_{i=0}^r (\lambda_{u_i, a_{u_i}} - d_1)} \right),$$

$$K_a = \begin{pmatrix} \begin{matrix} \lambda_{j_0, a_{j_0}^-} & \lambda_{j_1, a_{j_1}^-} & \dots & \lambda_{j_r, a_{j_r}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j_0, a_{j_0}^{r-1}} & \lambda_{j_1, a_{j_1}^{r-1}} & \dots & \lambda_{j_r, a_{j_r}^{r-1}} \\ \lambda_{j_0, a_{j_0}^r} & \lambda_{j_1, a_{j_1}^r} & \dots & \lambda_{j_r, a_{j_r}^r} \end{matrix} & \begin{matrix} \lambda_{t_0, a_{t_0}^-} & \lambda_{t_1, a_{t_1}^-} & \dots & \lambda_{t_r, a_{t_r}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{t_0, a_{t_0}^{r-1}} & \lambda_{t_1, a_{t_1}^{r-1}} & \dots & \lambda_{t_r, a_{t_r}^{r-1}} \\ \lambda_{t_0, a_{t_0}^r} & \lambda_{t_1, a_{t_1}^r} & \dots & \lambda_{t_r, a_{t_r}^r} \end{matrix} & \begin{matrix} \lambda_{u_0, a_{u_0}^-} & \lambda_{u_1, a_{u_1}^-} & \dots & \lambda_{u_r, a_{u_r}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{u_0, a_{u_0}^{r-1}} & \lambda_{u_1, a_{u_1}^{r-1}} & \dots & \lambda_{u_r, a_{u_r}^{r-1}} \\ \lambda_{u_0, a_{u_0}^r} & \lambda_{u_1, a_{u_1}^r} & \dots & \lambda_{u_r, a_{u_r}^r} \end{matrix} \\ \hline \begin{matrix} \lambda_{j_0, a_{j_0}^-} & \lambda_{j_1, a_{j_1}^-} & \dots & \lambda_{j_r, a_{j_r}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j_0, a_{j_0}^{r-1}} & \lambda_{j_1, a_{j_1}^{r-1}} & \dots & \lambda_{j_r, a_{j_r}^{r-1}} \\ \frac{\theta_i}{\lambda_{j_0, a_{j_0}^-} - d_0} & \frac{\theta_i}{\lambda_{j_1, a_{j_1}^-} - d_0} & \dots & \frac{\theta_i}{\lambda_{j_r, a_{j_r}^-} - d_0} \end{matrix} & \begin{matrix} \lambda_{t_0, a_{t_0}^-} & \lambda_{t_1, a_{t_1}^-} & \dots & \lambda_{t_r, a_{t_r}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{t_0, a_{t_0}^{r-1}} & \lambda_{t_1, a_{t_1}^{r-1}} & \dots & \lambda_{t_r, a_{t_r}^{r-1}} \\ \frac{\theta_k}{\lambda_{t_0, a_{t_0}^-} - d_0} & \frac{\theta_k}{\lambda_{t_1, a_{t_1}^-} - d_0} & \dots & \frac{\theta_k}{\lambda_{t_r, a_{t_r}^-} - d_0} \end{matrix} & \begin{matrix} \lambda_{u_0, a_{u_0}^-} & \lambda_{u_1, a_{u_1}^-} & \dots & \lambda_{u_r, a_{u_r}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{u_0, a_{u_0}^{r-1}} & \lambda_{u_1, a_{u_1}^{r-1}} & \dots & \lambda_{u_r, a_{u_r}^{r-1}} \\ \frac{\theta_l}{\lambda_{u_0, a_{u_0}^-} - d_0} & \frac{\theta_l}{\lambda_{u_1, a_{u_1}^-} - d_0} & \dots & \frac{\theta_l}{\lambda_{u_r, a_{u_r}^-} - d_0} \end{matrix} \\ \hline \begin{matrix} \lambda_{j_0, a_{j_0}^-} & \lambda_{j_1, a_{j_1}^-} & \dots & \lambda_{j_r, a_{j_r}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j_0, a_{j_0}^{r-1}} & \lambda_{j_1, a_{j_1}^{r-1}} & \dots & \lambda_{j_r, a_{j_r}^{r-1}} \\ \frac{\delta_i}{\lambda_{j_0, a_{j_0}^-} - d_1} & \frac{\delta_i}{\lambda_{j_1, a_{j_1}^-} - d_1} & \dots & \frac{\delta_i}{\lambda_{j_r, a_{j_r}^-} - d_1} \end{matrix} & \begin{matrix} \lambda_{t_0, a_{t_0}^-} & \lambda_{t_1, a_{t_1}^-} & \dots & \lambda_{t_r, a_{t_r}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{t_0, a_{t_0}^{r-1}} & \lambda_{t_1, a_{t_1}^{r-1}} & \dots & \lambda_{t_r, a_{t_r}^{r-1}} \\ \frac{\delta_k}{\lambda_{t_0, a_{t_0}^-} - d_1} & \frac{\delta_k}{\lambda_{t_1, a_{t_1}^-} - d_1} & \dots & \frac{\delta_k}{\lambda_{t_r, a_{t_r}^-} - d_1} \end{matrix} & \begin{matrix} \lambda_{u_0, a_{u_0}^-} & \lambda_{u_1, a_{u_1}^-} & \dots & \lambda_{u_r, a_{u_r}^-} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{u_0, a_{u_0}^{r-1}} & \lambda_{u_1, a_{u_1}^{r-1}} & \dots & \lambda_{u_r, a_{u_r}^{r-1}} \\ \frac{\delta_l}{\lambda_{u_0, a_{u_0}^-} - d_1} & \frac{\delta_l}{\lambda_{u_1, a_{u_1}^-} - d_1} & \dots & \frac{\delta_l}{\lambda_{u_r, a_{u_r}^-} - d_1} \end{matrix} \end{pmatrix} \neq 0. \tag{31}$$

which is a nonzero element in \mathbf{F}_{q_0} because $\frac{1}{\prod_{i=0}^{r-1} (\lambda_{t_i, a_{t_i}^-} - d_1)}$ and $\frac{1}{\prod_{i=0}^{r-1} (\lambda_{u_i, a_{u_i}^-} - d_1)}$ are in $G \subset \mathbf{F}_{q_0}$ while δ_k and δ_l in different cosets of G in $\mathbf{F}_{q_0} \setminus \{0\}$, and $\prod_{i=0}^{r-1} (\lambda_{j_i, a_{j_i}^-} - d_0) \in \mathbf{F}_{q_0} \setminus \{0\}$. It can be similarly proved that the coefficients of θ_k and θ_l are also in $\mathbf{F}_{q_0} \setminus \{0\}$. By C3, this linear combination, i.e., K'_a cannot be zero. By C1, Lemma 7, R1 and R2 of Lemma 6, we have that $D_0 D_1 D_2 \neq 0$. Therefore, $K_a \neq 0$ and $\det(B_a) \neq 0$, $\text{blkdiag}(B_0, B_1, \dots, B_{\ell-1})$ and thus \hat{H} is nonsingular, and the original file can be reconstructed.

This completes the proof. \square

V. CONCLUSION

In this paper, we proposed two constructions of PMDS array codes with two global parities and with $(1 + \epsilon)$ -optimal repair bandwidth, the required finite fields, and sub-packetization levels are much smaller than the one in [38]. The first one can support an arbitrary number of local parities and provide a tradeoff between the sub-packetization level and the repair bandwidth. In contrast, the other one is limited to two local parities but has smaller rebuilding access and its sub-packetization level is only 2. In addition, we presented an explicit PMDS array code with three global parities that has a smaller sub-packetization level as well as smaller repair bandwidth, the required finite field is also significantly smaller than the ones in [38].

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